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ABSTRACT

We consider the optimal stopping of a class of spectrally negative jump diffusions. We state a set of conditions under which the value is shown to have a representation in terms of an ordinary nonlinear programming problem. We establish a connection between the considered problem and a stopping problem of an associated continuous diffusion process and demonstrate how this connection may be applied for characterizing the stopping policy and its value. We also establish a set of typically satisfied conditions under which increased volatility as well as higher jump-intensity decelerates rational exercise by increasing the value and expanding the continuation region.

JEL Classification: C61, G11, G12

Keywords: jump diffusions, optimal stopping, nonlinear programming, perpetual American options

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1 Introduction

It is a well-known result from literature on mathematical finance that the price of a perpetual American option on an underlying asset whose value can be characterized as a stochastic process coincides with the value of an optimal stopping problem for this process (see, for example, Karatzas and Shreve (1999) pp. 54–87 and Øksendal (2003), pp. 290–298). Such option prices, while naturally of interest in themselves, can also be used as upper bounds for prices of American options with finite expiration dates. Thus, their role is of importance from a risk management point of view as well. Perpetual optimal stopping problems arise quite naturally also in the real options literature on the valuation of irreversible investment opportunities (see Dixit and Pindyck (1994) for an extensive textbook treatment of this theory). In that modeling framework the investment decision is usually interpreted as an opportunity (but not obligation) to obtain a stochastically fluctuating return in exchange from a payment (sunk cost) which may or may not be stochastic as well. Given the considerable planning horizon of the valuation of real investment opportunities, the time horizon is typically assumed to be infinite and, consequently, the considered optimal timing problem of the investment opportunity is assumed to be perpetual.

When the dynamics of the underlying process are characterizable via an Itô stochastic differential equation of form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{1}$$

with W a standard Wiener process, the stopping problem has been widely studied by relying on various techniques. The probably most usually applied approach is to rely on variational inequalities or the classical Hamilton-Jacobi-Bellman approach due to its applicability in a multidimensional setting as well (cf. Øksendal (2003) and Øksendal and Reikvam (1998)). In the one-dimensional setting there are, however, several different techniques for analyzing the perpetual stopping problem. The most general approach is probably provided by studies relying on the integral characterization of excessive functions for diffusion processes and the Martin boundary theory (cf. Salminen (1985) and Borodin and Salminen (2002), pp. 32–35). Alternatively, the considered problem can be analyzed by relying on the relationship between functional concavity and r-excessivity along the lines of the pioneering work by Dynkin (1965) (Chapters XV and XVI) and by Dynkin and Yuskevich (1969) which has been subsequently applied within a general optimal stopping framework in Dayanik and Karatzas (2003). A third technique for studying the perpetual optimal stopping in the linear diffusions setting is provided by the approaches relying on the well-known relationship between excessivity and superharmonicity with respect to first exit times from open sets with compact closure in the state space of the considered diffusion (cf. Dynkin (1965), Theorem 12.4). In such case, the optimal stopping problem is reduced to the optimization of arbitrary boundaries and, therefore, can be analyzed by relying on ordinary nonlinear programming techniques (cf. Alvarez (2001) and Alvarez (2004)).

More recently, the shortcomings of continuous models driven by Brownian motion have been discussed extensively and more general models allowing discontinuities have been studied to a considerable extent. The most simple generalization of the traditional continuous models is probably achieved by jump diffusion models, that is, by models allowing the driving noise to be a Lévy process. Lévy processes can be used to construct more realistic models of financial quantities, as they are able to accommodate jump discontinuities and the leptokurtic feature of return distributions, unlike the Gaussian models based on Brownian motion and normal distribution. For a taste of the aforementioned considerable amount of research on pricing American options and optimal stopping in Lévy models, see (for example) Gerber and Landry (1998), Gerber and Shiu (1998), Duffie et al (2000), Mordecki (2002a), Mordecki (2002b), Boyarchenko and Levendorskiĭ (2002), Alili and Kyprianou (2005) and Mordecki and Salminen (2006).

In risk management a criticism often leveled against the continuous models is their inability to model downside risk: the possibility of an instantaneous drop in the value of an asset. In real life markets phenomena closely resembling such instantaneous drops are often observed (for example, sudden unanticipated deterioration of stock market values, credit defaults, etc.). An empirically observed fact is that in the stock market reactions to negative shocks are usually significantly stronger than the reactions to positive ones (this is the celebrated "bad news" principle originally introduced in the seminal study by Bernanke (1983)). Hence, in light of this asymmetric nature of the reaction to unanticipated shocks, a prudent approach is to disregard possibilities for positive surprises and to take fully into account the possibilities for disadvantageous future occurrences. Consequently, a one-sided model that allows instantaneous downward jumps can be seen as a completely acceptable model from a prudent risk management point of view.

Motivated by our previous arguments, it is our objective in this study to consider a spectrally negative one-dimensional jump diffusion, say X, with a state space $I = (a, b) \subseteq \mathbb{R}$ and natural boundaries a and b. Interestingly, we establish that given some extra conditions on X, the value of the optimal stopping problem has a representation in terms of an ordinary nonlinear programming problem (cf. Alvarez (2001) and Alvarez (2003) for an associated result in the continuous diffusion case). This representation is valid for continuous, almost everywhere differentiable reward functions g satisfying the condition

$$g(x)/\psi(x) \text{ has a unique maximizer } x^* \in I \text{ and is non-increasing for } x > x^*,$$

where ψ is an increasing solution of the integro-differential equation $\mathcal{G}\psi = r\psi$ with $\mathcal{G}$ being the operator representing the infinitesimal generator of X. The representation is proved using the viscosity solution approach and, thus, smooth pasting may not necessarily hold. We find that given a jump diffusion for which the representation is valid in a certain class of reward functions, any strictly increasing C^2 transformation also has a similar representation, albeit for a different class of reward functions.

For the sake of comparison, we consider an optimal stopping problem of an associated continuous diffusion process which can be obtained by removing the pure jump part of the considered Lévy diffusion. We demonstrate that the value of the considered jump-diffusion stopping problem can be "sandwiched" between the values of two stopping problems which are defined with respect to the associated continuous diffusion. This finding is of interest since it can be applied for deriving bounds for the exercise threshold of the considered optimal stopping problem for the underlying jump-diffusion. Moreover, since the restricting values defined with respect to the continuous diffusion differ only by the rate at which they are discounted, our findings indicate that under some circumstances the downside jump-risk can be directly incorporated into the continuous diffusion case by adjusting the discount rate appropriately. This characterization is also important in the analysis of the impact of downside risk on the optimal stopping policy since according to this representation the optimal exercise boundary is lower for the underlying jump-diffusion than for the associated dominating continuous diffusion process provided that both valuations are discounted at the same rate.

We also consider the comparative static properties of the optimal stopping policy and its value and present a set of relatively general conditions under which the value of the considered problem is monotonic and convex. Along the lines of previous studies considering the optimal stopping of linear diffusions, we find that in such a case higher volatility increases the value of the optimal strategy and expands the continuation region where stopping is suboptimal by increasing the optimal exercise threshold. These observations are of interest since they indicate that higher volatility decelerates the rational exercise of investment opportunities by increasing the option value of waiting in the presence of jump diffusions as well. We also analyze the impact of increased jump-intensity on the optimal policy and its value and find that if the value is convex, then higher jump-intensity increases the value of waiting and decelerates rational exercise by expanding the continuation region. These observations emphasize the potentially significant combined negative effect of jump-risk and continuous systematic risk on the timing of irreversible investment policies.

The contents of this study are as follows. In section 2, we present the model and the assumptions used throughout the study. The representation of the stopping problem in terms of an ordinary optimization problem, is stated and proved in section 3, together with the result on the validity of the representation for increasing C^2 transforms of Lévy diffusions admitting the representation. Some useful inequalities related to the associated continuous diffusion are presented in section 4, and Section 5 is devoted to comparative statics. Explicit illustrations are given in section 6, and section 7 concludes.

2 The Setup and Basic Assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying a standard Wiener process $W = \{W_t\}$ and a compound Poisson process $J = \{J_t\}$ with intensity λ and some jump size distribution. We can define a finite activity Lévy process $L = \{L_t\}$ by

$$L_t = t + W_t + J_t. \tag{2}$$

We equip $(\Omega, \mathcal{F}, \mathbb{P})$ with the completed natural filtration $\mathbb{F}$ generated by this process. The natural filtration of a Lévy process is right-continuous, and thus the completed filtration satisfies the usual hypotheses (see Protter (2004) Theorem I.31). We consider the optimal stopping problem

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} g(X_\tau) \right\}, \tag{3}$$

where $X = \{X_t\}$ is the jump diffusion driven by L with initial value $X_0 = x \in I$ and dynamics given by the stochastic differential equation

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t + \int_{\mathcal{S}(m)} \gamma(X_t, z)\tilde{N}(dz, dt). \tag{4}$$

In the above equations $\tilde{N}(U, t)$ is a compensated Poisson random measure, $\mathcal{S}(m) \subset (0, \infty)$ is the support of the corresponding Lévy measure m and $\mathcal{T}$ is the set of all $\mathbb{F}$ -stopping times. Note that the driving jump process is, as a compensated compound Poisson process, a martingale – this is no restriction, as non-martingale jump dynamics can be reduced to the form 4 by adding and subtracting a correction term on the left hand side of the stochastic differential equation. We denote the expectation of the jump size by $\overline{m}$. The state space of the Lévy diffusion is an open interval $I := (a, b) \subseteq \mathbb{R}$ where a and b are natural boundaries (not attainable in finite time). We assume that the coefficient functions in 4 satisfy the usual conditions for the existence of a unique adapted càdlàg solution $X \in L^2(\mathbb{P})$ without explosions. In the case of an infinite interval I, sufficient conditions are at most linear growth and Lipschitz continuity, see Øksendal and Sulem (2005) Theorem 1.19. The global Lipschitz condition guarantees that the explosion time of the process is a.s. infinite (see Protter (2004) Theorem V.40). In addition, we assume that the coefficients have locally Lipschitz first derivatives.

The solution of the optimal stopping problem is known to be closely related to the integro-differential equation defined for $f \in C_0^2(\mathbb{R})$ by

$$\mathcal{G}f = rf, \tag{5}$$

where $(\mathcal{G}f)(x)$ is the generator of X given by

$$\frac{1}{2}\sigma^2(x)f''(x) + \alpha(x)f'(x) + \lambda \int_{\mathcal{S}(m)} \{f(x+\gamma(x,z)) - f(x) - f'(x)\gamma(x,z)\}\, m(dz). \tag{6}$$

Integrating the last two terms of the integrand in (6) and using the notation $(\mathcal{G}_r) := (\mathcal{G} - r)$ we can write (5) equivalently as

$$(\mathcal{G}_r f)(x) = \frac{1}{2}\sigma^2(x)f''(x) + \tilde{\alpha}(x)f'(x) - \tilde{r}f(x) + \lambda \int_{\mathcal{S}(m)} f(x+\gamma(x,z))m(dz) = 0, \tag{7}$$

where $\tilde{\alpha}(x) = \alpha(x) - \lambda \int_{\mathcal{S}(m)} \gamma(x,z)m(dz)$ and $\tilde{r} = r + \lambda$.

Next the assumptions used throughout the rest of this study are stated. We denote by τ_S the first exit time of the process X from an open set $S \subset \mathbb{R}$. The following additional assumptions concerning the dynamics of X are made:

X1. $\tau_{(a,\overline{x})} = \inf\{t \geq 0 \,:\, X_t \geq \overline{x}\} < \infty$ $\mathbb{P}_x$-a.s. for all $a < x < \overline{x} < b$;

X2. $a - x < \gamma(x,z) \leq 0$ for all $(x,z) \in I \times \mathcal{S}(m)$.

Assumption *X2* implies that X has only negative jumps and that X cannot reach the lower boundary a by jumping. Thus $X_t \in I$ for all $t \geq 0$.

The reward function g is assumed to satisfy

g1. $g(x) = \max(\tilde{g}(x), 0)$ with $\tilde{g}$ increasing, continuous and C^2 on $I \setminus N$ for some finite set $N \subset I$ with finite limits $\tilde{g}'(x+)$, $\tilde{g}''(x+)$ for $x \in N$, and such that $\tilde{g}(a) \leq 0$.

Note that assumption *g1* is satisfied by the reward of a standard American call option, in which case $\tilde{g}(x) = x - K$ for some strike price K. In fact, the imposed reward structure is natural for an option type contract, where we can always avoid losses by not exercising our option if the reward is negative.

We make the following assumption on the operator $\mathcal{G}_r$:

A1. $\mathcal{G}_r\psi = 0$ has an increasing solution $\psi \in C^2(I)$ such that $\psi(a) = 0$.

Noteworthy is that it is not at all clear whether a given integro-differential equation has an increasing solution – the validity of assumption *A1* needs to be carefully checked in each case. Finally, we need to make two assumptions on the behavior of the quotient g/ψ, namely,

Ag1. there exists a unique maximizer $x^* \in I$ of $g(x)/\psi(x)$ and $g(x)/\psi(x)$ is non-increasing for $x > x^*$.

Ag2. there exists $\hat{x} < x^*$ such that, for all $x \geq \hat{x}$ such that g is C^2 at x,

$$(\mathcal{G}_r g)(x) \leq -\int_{C(z)} \left\{ \frac{g(x^*)}{\psi(x^*)}\psi(x + \gamma(x,z)) - g(x + \gamma(x,z)) \right\} \nu(dz),$$

where $C(z) = \big\{ z \in \mathcal{S}(m) \,:\, x + \gamma(x,z) < x^* \big\}$.

In a sense, the last assumption is needed to guarantee the r-excessivity of the value function V in the stopping region $x \geq x^*$, as will be seen in the proof of theorem 3.3 later on. In most cases, this assumption is rather difficult to verify otherwise than numerically on a case by case basis. It should be noted that assumptions *Ag1* and *Ag2* have implications for the form of the reward function g: the set of allowable reward functions will depend on the behavior of function ψ.

3 The Representation Theorem

In Alvarez (2001), it is shown that (modulo some conditions) if the process X is a continuous linear diffusion the value function of the stopping problem (3) can be expressed as

$$V(x) = \psi(x) \sup_{y \geq x} \left\{ \frac{g(y)}{\psi(y)} \right\},$$

where $\psi(x)$ is the increasing fundamental solution of the differential equation $A\psi - r\psi = 0$, where A is the second order differential operator coinciding with the infinitesimal generator of X. Our main theorem states that this representation is also valid for a jump diffusion, provided that the assumptions of section 2 are satisfied. Before stating the main result, we present some auxiliary results necessary for the proof of the theorem. At this point, we introduce the notation $v(x) := \psi(x) \sup_{y \geq x} \left\{ \frac{g(y)}{\psi(y)} \right\}$ and consider the properties of this function.

Lemma 3.1. *Assume that g/ψ is continuous on I with a unique maximum point $x^* \in I$ and non-increasing for $x > x^*$. Then $v(x)$ is a continuous function of x and we have the representation*

$$v(x) = \begin{cases} g(x), & x \geq x^* \\ \psi(x)\frac{g(x^*)}{\psi(x^*)}, & x < x^*. \end{cases}$$

Proof. For a function $f := g/\psi$ satisfying the assumptions,

$$\sup_{y \geq x} f(y) = \begin{cases} f(x), & x \geq x^* \\ f(x^*), & x < x^*, \end{cases}$$

which is continuous if f is. The representation is immediate (multiply the above equation with $\psi(x)$). $\square$

Lemma 3.1 demonstrates, that under our assumptions the value of the associated nonlinear programming problem is continuous. Interestingly, as in studies based on continuous diffusion models, lemma 3.1 characterizes the value in terms of the exercise payoff received at the exercise boundary and the ratio $\psi(x)/\psi(x^\star)$ measuring the expected present value of a contract which pays the holder one dollar at the first date the underlying jump diffusion exceeds a beforehand fixed threshold level. This observation is expressed in more precise terms in the following lemma.

Lemma 3.2. *Suppose $\psi : I \mapsto \mathbb{R}_+$ is an increasing solution of $\mathcal{G}_r u = 0$ and $a < x < y < b$. Then*

$$\mathbb{E}_x[e^{-r\tau_{(a,y)}}] = \frac{\psi(x)}{\psi(y)}.$$

Moreover, in case $\psi(x)$ exists any other nonnegative and increasing solution of $\mathcal{G}_r u = 0$ is a constant multiple of $\psi(x)$ (i.e. $\psi(x)$ is unique up to a multiplicative constant).

Proof. Under assumption *X1* and the assumed boundary behavior of X_t at the boundary a, we can apply the Dynkin formula to ψ:

$$\mathbb{E}_x[e^{-r\tau_{(a,y)}}\psi(X_{\tau_{(a,y)}})] = \psi(x) + \mathbb{E}_x \int_0^{\tau_{(a,y)}} e^{-rt}(\mathcal{G}_r\psi)(X_t)dt.$$

Since ψ solves $\mathcal{G}_r\psi = 0$ and $X_{\tau_{(a,y)}} = y$ a.s. (because X has no positive jumps and it never attains a), this implies that

$$\psi(y)\mathbb{E}_x[e^{-r\tau_{(a,y)}}] = \psi(x),$$

from which the first result follows. To establish uniqueness, assume that $\varsigma : I \mapsto \mathbb{R}_+$ is another increasing and nonnegative solution of equation $\mathcal{G}_r u = 0$. By applying a similar argument as above, we find that

$$\varsigma(x) = \frac{\varsigma(y)}{\psi(y)}\psi(x)$$

which completes the proof of our lemma. □

It is worth emphasizing that the strong Markov property of the jump diffusion and the fact that it cannot jump upwards and, therefore, that it can increase only continuously imply that the function $\mathbb{E}_x[e^{-r\tau_{(a,y)}}]$ can always be expressed as a ratio of the form (8). However, it is not beforehand clear whether this ratio is always (i.e. for any jump diffusion model) twice continuously differentiable with respect to the current state or not. Hence, lemma 3.2 essentially demonstrates that in those cases where the integro-differential equation $\mathcal{G}_r u = 0$ has an increasing solution, the expected value $\mathbb{E}_x[e^{-r\tau_{(a,y)}}]$ can be expressed in terms of this solution and identity (8) holds. The key implication of this finding and our main result on the characterization of the value of the considered optimal stopping problem as an ordinary nonlinear programming problem is now summarized in the following.

Theorem 3.3. *Suppose X and g are such that* X1–X2, g1, A1, Ag1 *and* Ag2 *are satisfied. Then, if the value function of problem* (3) *is continuous, it has* *the representation*

$$V(x) = \psi(x) \sup_{y \geq x} \left\{ \frac{g(y)}{\psi(y)} \right\}, \tag{8}$$

where ψ is an increasing solution of $\mathcal{G}_r \psi = 0$.

Proof. We use again the notation $v(x) := \psi(x) \sup_{y \geq x} \left\{ \frac{g(y)}{\psi(y)} \right\}$ and begin by proving an auxiliary result $(\mathcal{G}_r v)(x) = 0$ for $x < x^*$ (where x^* is the unique maximizer of $g(x)/\psi(x)$) via the following direct calculation:

$$\tfrac{1}{2}\sigma^2(x) v''(x) + \tilde{\alpha}(x) v'(x) - \tilde{r} v(x) + \lambda \int_{\mathcal{S}(m)} v(x + \gamma(x,z)) m(dz)$$

$$= \tfrac{g(x^*)}{\psi(x^*)} \left[\tfrac{1}{2}\sigma^2(x)\psi''(x) + \tilde{\alpha}(x)\psi'(x) - \tilde{r}\psi(x) \right] + \lambda \int_{\mathcal{S}(m)} v(x + \gamma(x,z)) m(dz)$$

$$= -\tfrac{g(x^*)}{\psi(x^*)} \cdot \lambda \int_{\mathcal{S}(m)} \psi(x + \gamma(x,z)) m(dz) + \lambda \int_{\mathcal{S}(m)} v(x + \gamma(x,z)) m(dz)$$

$$= \lambda \int_{\{x+\gamma(x,z) < x^*\}} \tfrac{g(x^*)}{\psi(x^*)} \big(-\psi(x + \gamma(x,z)) + \psi(x + \gamma(x,z)) \big) m(dz) +$$

$$+ \lambda \int_{\{x+\gamma(x,z) > x^*\}} \left[-\tfrac{g(x^*)}{\psi(x^*)} \psi(x + \gamma(x,z)) + g(x + \gamma(x,z)) \right] m(dz).$$

For a process with $\gamma(x,z) \leq 0$ the second integral in the last expression vanishes, and the first integrand is identically zero. Auxiliary result is now proved.

Consider an increasing sequence $\{\overline{x}_N\}_{N \in \mathbb{N}} \subset I$ such that $\overline{x}_1 > x^*$ and $\overline{x}_N \to b$. Denote $\tau_N = \tau_{(a,\overline{x}_N)}$. If w is a continuous viscosity solution of the variational inequalities

$$\max \big((\mathcal{G}_r w)(x), g(x) - w(x) \big) = 0, \; x \in (a, \overline{x}_N), \tag{9}$$

satisfying the boundary conditions

$$w(a) = g(a), \; w(\overline{x}_N) = g(\overline{x}_N), \tag{10}$$

then by virtue of theorem 9.4 in Øksendal and Sulem (2005) $w(x) = V_N(x) := \sup_{\tau \leq \tau_N} \mathbb{E}_x \left[e^{-r\tau} g(X_\tau) \right]$ for all $x \in [a, \overline{x}_N]$. Note that the uniform integrability condition in that theorem is needed to ascertain that $w(x)$ is indeed attainable (see Øksendal and Reikvam (1998)). In our case this condition need not be imposed, as by lemma 3.2 we have, for $x < x^\star$,

$$v(x) = g(x^*)\frac{\psi(x)}{\psi(x^*)} = \mathbb{E}_x[e^{-r\tau_{(a,x^*)}}g(X_{\tau_{(a,x^*)}})],$$

and for $x \geq x^\star$, $v(x) = g(x)$. Now we show that the function v (or, more precisely, its restriction to $I_N := (a, \overline{x}_N)$) is a continuous viscosity solution of 9 such that 10 is satisfied, for any $N \in \mathbb{N}$.

Since v is continuous by lemma 3.1, it remains to show that v is a solution of the variational inequality in the viscosity sense. First we establish the *subsolution property.* So let us take $x_0 \in I_N$ and suppose that $h \in C^2(I_N)$ is such that $h(x) \geq v(x)$ for $x \in I_N$ and $h(x_0) = v(x_0)$. We have two possibilities:

(i) if $a < x_0 < x^*$, then $h(x) - v(x)$ is a smooth function at $x = x_0$ and has a local minimum there. First and second order conditions for a local minimum imply then that $v'(x_0) = h'(x_0)$ and $v''(x_0) \leq h''(x_0)$. Furthermore, $h(x_0 + \gamma(x_0, z)) \geq v(x_0 + \gamma(x_0, z))$. But then $(\mathcal{G}_r h)(x_0) \geq (\mathcal{G}_r v)(x_0) = 0$, and the variational inequality

$$\max\left((\mathcal{G}_r h)(x_0), g(x_0) - v(x_0)\right) \geq 0, \tag{11}$$

holds.

(ii) if $\overline{x}_N > x_0 \geq x^*$, then $v(x_0) = g(x_0)$ and 11 is satisfied.

Thus, for all $h \in C^2(I_N)$ and $x_0 \in I_N$ such that $h(x) \geq v(x)$, for $x \in I_N$, and $h(x_0) = v(x_0)$, equation 11 is satisfied, so v is a viscosity subsolution of the variational inequality.

To show the *supersolution property* of v we take $x_0 \in I_N$ and $h \in C^2(I_N)$ such that $h(x) \leq v(x)$ for all $x \in I_N$ and $h(x_0) = v(x_0)$. Now we have three possibilities:

(i) if $a < x_0 < x^*$, then $h(x) - v(x)$ is a smooth function at $x = x_0$ and has a local maximum there. First and second order conditions for a local maximum imply then that $v'(x_0) = h'(x_0)$ and $v''(x_0) \geq h''(x_0)$. Furthermore, $h(x_0 + \gamma(x_0, z)) \leq v(x_0 + \gamma(x_0, z))$. But then $(\mathcal{G}_r h)(x_0) \leq (\mathcal{G}_r v)(x_0) = 0$, and since $v(x_0) \geq g(x_0)$, the inequality

$$\max\left((\mathcal{G}_r h)(x_0), g(x_0) - v(x_0)\right) \leq 0, \tag{12}$$

is satisfied.

(ii) if $\overline{x}_N > x_0 \geq x^*$ and g is C^2 at x_0, then $v(x_0) = g(x_0)$ and so the second half of the left hand side of 12 equals 0. To obtain $(\mathcal{G}_r h)(x_0) \leq 0$, observe that by arguments similar to previous ones, $(\mathcal{G}_r h)(x_0) \leq (\mathcal{G}_r v)(x_0)$, and from the definition of v we get then, using assumption *Ag2*,

$$(\mathcal{G}_r v)(x_0) = (\mathcal{G}_r g)(x_0) + \int_{C(z)} \{v(x_0+\gamma(x_0,z)) - g(x_0+\gamma(x_0,z))\}\nu(dz) \leq 0$$

(see section 2 for the definition of the set $C(z)$). This implies that 12 holds.

(iii) if $\overline{x}_N > x_0 \geq x^*$ and $x_0 \in N$ (i.e. g is not C^2 at x_0), we still have $g(x_0) = v(x_0)$. By (ii), under our assumptions

$$(\mathcal{G}_r h)(y) \leq (\mathcal{G}_r g)(y) + \int_{C(z)} \{v(y+\gamma(y,z)) - g(y+\gamma(y,z))\}\nu(dz) \leq 0, \quad (13)$$

for all $x_0 < y < \min\{\overline{x}_N, x_k\}$, where x_k is the point of $N \cap (x_0, \overline{x}_N)$ closest to x_0 . Since g is C^2 on (x_0, x_k) and v is continuous, denoting $\lim_{y\downarrow x_0} y = \tilde{x}$, we get

$$(\mathcal{G}_r h)(x_0) \leq (\mathcal{G}_r g)(\tilde{x}) + \int_{C(z)} \{v(\tilde{x}+\gamma(\tilde{x},z)) - g(\tilde{x}+\gamma(\tilde{x},z)\}\nu(dz) \leq 0. \quad (14)$$

So 12 holds.

We have established that for all $h \in C^2(I_N)$ and $x_0 \in I$ such that $h(x) \geq v(x)$, for $x \in I_N$, and $h(x_0) = v(x_0)$ equation 12 holds, i.e. v is a viscosity supersolution of the variational inequality.

We have now proved that being continuous and both a viscosity sub- and supersolution, v is a continuous viscosity solution of the variational inequalities. Since by the definition of v and assumption *A1* the boundary conditions 10 are satisfied, the uniqueness result of Theorem 9.4 in Øksendal and Sulem (2005) implies that $v(x) = V_N(x)$ on I_N for any $N \in \mathbb{N}$. On the other hand,

$$V_N(x) = \sup_{\tau \leq \tau_N} \mathbb{E}_x\left[e^{-r\tau} g(X_\tau)\right] \to \sup_{\tau \leq \infty} \mathbb{E}_x\left[e^{-r\tau} g(X_\tau)\right] = V(x) \quad (15)$$

as $N \to \infty$. Thus the increasing sequence of functions $\{V_N\} = \{v|_{I_N}\}$ converges to V as $N \to \infty$ for any $x \in I$ such that $V(x)$ is finite. Thus $v(x) = V(x)$.

□

The representation of theorem 3.3 implies that if g is continuously differentiable at the point x^* maximizing $g(x)/\psi(x)$, then x^* can be solved from the first order condition for an extremum $g'(x)\psi(x) - g(x)\psi'(x) = 0$, which is equivalent to the logarithmic derivative condition $D_x[\ln g(x)] = D_x[\ln \psi(x)]$. In this case the well-known *smooth fit condition* is satisfied, i.e. the value is continuously differentiable. However, even if x^* happens to be a point of nondifferentiability of g, the representation result holds – necessary conditions for a maximum of g/ψ are then

$$\lim_{y \to x^*-} \{g'(y)\psi(y) - g(y)\psi'(y)\} \geq 0 \text{ and } \lim_{y \to x^*+} \{g'(y)\psi(y) - g(y)\psi'(y)\} \leq 0,$$

which imply only that

$$g'(x^*-) \geq V'(x^*-) = \psi'(x^*)\frac{g(x^*)}{\psi(x^*)} \geq g'(x^*+) = V'(x^*+).$$

It is possible to prove that given a process X such that theorem 3.3 holds (for a certain class of reward functions) and any sufficiently regular transformation $f(\cdot)$, the representation is valid for the process Y defined by $Y_t = f(X_t)$ (although the class of allowable reward functions will be different). This is the content of the next theorem.

Theorem 3.4. *Let $\{X_t\}$ be a stochastic process such that assumptions* X1, X2 *and* A1 *are satisfied, and let f be a strictly increasing function in $C^2(I)$. Denote the increasing solution in* A1 *for X by ψ_1. Define a new process Y by setting $Y_t := f(X_t)$. Then Y satisfies assumptions* X1, X2 *and* A1. *Furthermore, the corresponding increasing solution in* A1 *for Y is given by $\psi_1(f^{-1}(y))$.*

Proof. A C^2 transform of a jump diffusion is a jump diffusion. Being an increasing function, f maps the state space $I = (a, b)$ of X onto $J = (f(a), f(b))$, the state space of Y. Since

$$Y_t = f(X_t) > \overline{x} \Leftrightarrow X_t > f^{-1}(\overline{x}),$$

it follows that Y satisfies *X1*, and as X is spectrally negative and f is increasing, we have

$$\begin{aligned} |\Delta Y_t| &= |f(X_{t-}) - f(X_{t-} + \Delta X_t)| \\ &= f(X_{t-}) - f(X_{t-} + \Delta X_t) < f(X_{t-}) - f(a) \end{aligned}$$

and thus *X2* is satisfied. Because X is assumed to satisfy *A1*, there exists an increasing solution ψ_1 of the integro-differential equation

$$\tilde{\mu}(x)\psi'(x) + \frac{1}{2}\sigma^2(x)\psi''(x) + \int_{\mathcal{S}(m)} \psi(x+\gamma(x,z))\nu(dz) = \tilde{r}\psi(x).$$

Let $x := x(y) = f^{-1}(y)$. The transformation $\tilde{\psi}(y) = \psi(x) = (\psi \circ f^{-1})(y)$ leads to the integro-differential equation

$$\begin{aligned} &\tilde{\mu}(x(y))[x'(y)]^{-1}\tilde{\psi}'(y) + \tfrac{1}{2}\sigma^2(x(y))\Big\{[x'(y)]^{-2}\tilde{\psi}''(y) - \tilde{\psi}'(y)x''(y)[x'(y)]^{-3}\Big\} + \\ &+ \textstyle\int_{\mathcal{S}(m)} \tilde{\psi}(x(y)+\gamma(x(y),z))\nu(dz) = \tilde{r}\tilde{\psi}(y), \end{aligned} \tag{16}$$

which is well-defined since under the assumptions on f, the inverse mapping theorem guarantees the existence and continuity of $x'(y)$ and $x''(y)$. Defining

$$\phi(y) := \psi_1(x(y))$$

we can by substituting ϕ into (16) establish that $\phi(y)$ is a solution of (16). As ψ_1 is increasing on I by assumption and $x'(y) = (f^{-1})'(y) = (f'(x))^{-1} > 0$ on J by the inverse mapping theorem, we have that

$$\phi'(y) = \psi_1(x(y))x'(y) > 0.$$

So $\phi(y)$ is an increasing function, and $\phi(f(a)) = \psi_1(a) = 0$, since X satisfies *A1*. Thus Y satisfies *A1*. □

4 Useful Inequalities: Sandwiching the Solution

In this section we plan to analyze how the considered stopping problem is related to two optimal stopping problems of an associated continuous diffusion model. To accomplish this task, consider now the associated diffusion

$$d\tilde{X}_t := \left(\mu(\tilde{X}_t) - \int_{\mathcal{S}(m)} \gamma(\tilde{X}_t, z)\nu(dz)\right)dt + \sigma(\tilde{X}_t)dW_t. \tag{17}$$

It is worth mentioning that this associated diffusion is very useful in assessing the impact of downside risk on the optimal policy, as the Lévy diffusion X is, in fact, a superposition of $\tilde{X}$ and a spectrally negative, non-martingale jump process. As usually, we denote as $\tilde{\mathcal{A}}_\theta$ the differential operator

$$\tilde{\mathcal{A}}_\theta = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \left(\mu(x) - \int_{\mathcal{S}(m)} \gamma(x,z)\nu(dz)\right)\frac{d}{dx} - \theta \tag{18}$$

associated with the continuous diffusion $\tilde{X}_t$ killed at the constant rate $\theta > 0$. Along the lines of the notation in our previous analysis, we denote as $\tilde{\psi}_\theta(x)$ the increasing fundamental solution (i.e. the minimal increasing θ-harmonic mapping for the diffusion $\{\tilde{X}_t; t \geq 0\}$; for a thorough characterization of these mappings, see Borodin and Salminen (2002), p. 33) of the ordinary linear second order differential equation $(\tilde{\mathcal{A}}_\theta u)(x) = 0$. As is well-known from the classical theory of diffusions, given this increasing fundamental solution we have for all $x \leq y$ (cf. Borodin and Salminen (2002), p. 18)

$$\mathbb{E}_x \left[e^{-\theta \tilde{\tau}_{(a,y)}} \right] = \frac{\tilde{\psi}_\theta(x)}{\tilde{\psi}_\theta(y)},$$

where $\tilde{\tau}_{(a,y)} = \inf\{t \geq 0 : \tilde{X}_t = y\}$ denotes the first hitting time of the diffusion $\tilde{X}_t$ to the state y. Therefore, the continuity of the exercise payoff yields that for all $x \leq y$ we have

$$\mathbb{E}_x \left[e^{-\theta \tilde{\tau}_{(a,y)}} g(\tilde{X}_{\tilde{\tau}_{(a,y)}}) \right] = g(y) \frac{\tilde{\psi}_\theta(x)}{\tilde{\psi}_\theta(y)}$$

implying that

$$\sup_{y \geq x} \mathbb{E}_x \left[e^{-\theta \tilde{\tau}_{(a,y)}} g(\tilde{X}_{\tilde{\tau}_{(a,y)}}) \right] = \tilde{\psi}_\theta(x) \sup_{y \geq x} \left[\frac{g(y)}{\tilde{\psi}_\theta(y)} \right]$$

provided that the supremum exists. In light of this observation it is naturally of interest to ask whether the discount rate θ can be chosen so as to yield representations which either dominate or are smaller that the value of the optimal stopping problem (3). Interestingly, the answer to this question turns out to be positive as is illustrated by our following theorem characterizing the relationship of the value of the optimal stopping problem with the values of two associated stopping problems defined with respect to the continuous diffusion (17).

Theorem 4.1. *For all $x \leq y$ we have*

$$\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}(y)} \leq \mathbb{E}_x \left[e^{-r\tau_{(a,y)}} \right] \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r(y)}.$$

Consequently,

$$\tilde{\psi}_{r+\lambda}(x) \sup_{y \geq x} \left[\frac{g(y)}{\tilde{\psi}_{r+\lambda}(y)} \right] \leq \sup_{y \geq x} \mathbb{E}_x \left[e^{-r\tau_{(a,y)}} g(X_{\tau_{(a,y)}}) \right] \leq \tilde{\psi}_r(x) \sup_{y \geq x} \left[\frac{g(y)}{\tilde{\psi}_r(y)} \right]$$

provided that the supremum exists. Therefore, if condition A1 is satisfied, then

$$\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}(y)} \leq \frac{\psi(x)}{\psi(y)} \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r(y)}$$

for all $x \leq y$ and

$$\tilde{\psi}_{r+\lambda}(x) \sup_{y \geq x} \left[\frac{g(y)}{\tilde{\psi}_{r+\lambda}(y)}\right] \leq \psi(x) \sup_{y \geq x} \left[\frac{g(y)}{\psi(y)}\right] \leq \tilde{\psi}_r(x) \sup_{y \geq x} \left[\frac{g(y)}{\tilde{\psi}_r(y)}\right],$$

provided that the supremum exists.

Proof. Applying the Dynkin formula to $\tilde{\psi}_r(x)$ yields

$$\mathbb{E}_x[e^{-r\tau_{(a,y)}}\tilde{\psi}_r(X_{\tau_{(a,y)}})] = \tilde{\psi}_r(x) + \mathbb{E}_x \int_0^{\tau_{(a,y)}} e^{-rt}(\mathcal{G}_r\tilde{\psi}_r)(X_t)dt \leq \tilde{\psi}_r(x)$$

since

$$(\mathcal{G}_r\tilde{\psi}_r)(x) = \lambda \int_{\mathcal{S}(m)} \{\tilde{\psi}_r(x + \gamma(x,z)) - \tilde{\psi}_r(x)\}m(dz) < 0$$

by the monotonicity of $\tilde{\psi}_r(x)$. Since $X_{\tau_{(a,y)}} = y$ a.s. (because X has no positive jumps and it never attains a) and $\tilde{\psi}_r(x)$ is continuous, this inequality implies that $\mathbb{E}_x[e^{-r\tau_{(a,y)}}] \leq \tilde{\psi}_r(x)/\tilde{\psi}_r(y)$ for all $x \leq y$. Analogously, applying the Dynkin formula to $\tilde{\psi}_{r+\lambda}(x)$ yields

$$\mathbb{E}_x[e^{-r\tau_{(a,y)}}\tilde{\psi}_{r+\lambda}(X_{\tau_{(a,y)}})] = \tilde{\psi}_{r+\lambda}(x) + \mathbb{E}_x \int_0^{\tau_{(a,y)}} e^{-rt}(\mathcal{G}_r\tilde{\psi}_{r+\lambda})(X_t)dt \geq \tilde{\psi}_{r+\lambda}(x)$$

since

$$(\mathcal{G}_r\tilde{\psi}_{r+\lambda})(x) = \lambda \int_{\mathcal{S}(m)} \tilde{\psi}_{r+\lambda}(x + \gamma(x,z))m(dz) > 0$$

by the positivity of $\tilde{\psi}_{r+\lambda}(x)$. Thus, we observe that $\mathbb{E}_x[e^{-r\tau_{(a,y)}}] \geq \tilde{\psi}_{r+\lambda}(x)/\tilde{\psi}_{r+\lambda}(y)$ for all $x \leq y$. The rest of the alleged results then follow from the nonnegativity of $g(x)$ and condition A1. □

Theorem 4.1 essentially establishes that in the present setting the value of the optimal stopping problem (3) satisfies the inequality $\tilde{V}_{r+\lambda}(x) \leq V(x) \leq \tilde{V}_r(x)$, where

$$\tilde{V}_\theta(x) = \sup_\tau \mathbb{E}_x\left[e^{-\theta\tau}g(\tilde{X}_\tau)\right],$$

provided that the sufficiency conditions guaranteeing the optimality of the stopping rule characterized by a single threshold are satisfied. For that class of problems, Theorem 4.1 also clearly indicates that $C_{r+\lambda} \subseteq C \subseteq C_r$ where

$C_\theta = \{x \in I : \tilde{V}_\theta(x) > g(x)\}$ and $C = \{x \in I : V(x) > g(x)\}$. This observation is important since it demonstrates that the optimal exercise threshold x^* is dominated by the exercise threshold of the dominating value $\tilde{V}_r(x)$ and is greater than the exercise threshold of the smaller value $\tilde{V}_{r+\lambda}(x)$. In this way, the findings of our Theorem 4.1 provide valuable information on the impact of pure (uncompensated) down-side risk on the optimal decision. Moreover, since $\tilde{V}_{r+\lambda}(x) \leq \tilde{V}_\theta(x) \leq \tilde{V}_r(x)$ for all $x \in I$, we immediately observe that if the conditions of our Theorem 4.1 and Theorem 3.3 are satisfied, then there is a critical discount rate for which the stopping rule coincides in the continuous and in the jump-diffusion case. That is, there is a $\theta^* \in (r, r+\lambda)$ such that $x^* = \min\{x \in I : \tilde{V}_{\theta^*}(x) = g(x)\}$.

It is worth emphasizing that the proof of Theorem 4.1 essentially relies on the fact that if $u : I \mapsto \mathbb{R}_+$ is a sufficiently smooth and monotonically increasing function, then

$$(\tilde{\mathcal{A}}_{r+\lambda}u)(x) \leq (\mathcal{G}_r u)(x) \leq (\tilde{\mathcal{A}}_r u)(x).$$

Hence, our results clearly indicate that the class of sufficiently smooth monotonically increasing r-excessive mappings for the diffusion $\tilde{X}_t$ is larger than the class of sufficiently smooth monotonically increasing r-excessive mappings for the jump-diffusion X_t which, in turn, is larger than the class of sufficiently smooth monotonically increasing $(r+\lambda)$-excessive mappings for the diffusion $\tilde{X}_t$. This observation is interesting since it directly generates a natural ordering for the monotone (viscosity) solutions of the variational inequalities $\max\{(\mathcal{G}_r u)(x), g(x) - u(x)\} = 0$ and $\max\{(\tilde{\mathcal{A}}_\theta u)(x), g(x) - u(x)\} = 0$ with $\theta = r, r+\lambda$.

5 Comparative Statics

In this section our main objective is to consider comparative static properties of the value function and the optimal policy and, especially, to analyze the impact of increased volatility on these factors. To this end, we consider two jump diffusions of the form (4), X and $\hat{X}$, which are otherwise identical but have different volatilities, $\sigma(x) > \hat{\sigma}(x)$. In accordance with this notation, we denote the values of the associated optimal stopping problems by V and $\hat{V}$, the associated differential operators as $\mathcal{G}_r$ and $\hat{\mathcal{G}}_r$, and the associated increasing fundamental solutions (given that assumption A1 is satisfied) as ψ and $\hat{\psi}$, respectively. Our first result emphasizing the role of these fundamental solutions is now summarized in the next theorem.

Theorem 5.1. *Assume that the increasing fundamental solution $\psi(x)$ is convex. Then*

$$\frac{\hat{\psi}(x)}{\hat{\psi}(y)} \leq \frac{\psi(x)}{\psi(y)}$$

for all $x \leq y$. Hence,

$$\hat{\psi}(x) \sup_{y \geq x} \left[\frac{g(y)}{\hat{\psi}(y)} \right] \leq \psi(x) \sup_{y \geq x} \left[\frac{g(y)}{\psi(y)} \right]$$

provided that the supremum exists. Moreover, if the conditions of Theorem 3.3 are satisfied, then $V(x) \geq \hat{V}(x)$ and, therefore,

$$\hat{C} = \{x \in I : \hat{V}(x) > g(x)\} \subseteq \{x \in I : V(x) > g(x)\} = C.$$

If the increasing fundamental solution $\hat{\psi}(x)$ is concave, then the inequalities and inclusions stated above are reversed.

Proof. Applying the Dynkin formula to $\psi(x)$ yields

$$\mathbb{E}_x[e^{-r\hat{\tau}_{(a,y)}} \psi(\hat{X}_{\hat{\tau}_{(a,y)}})] = \psi(x) + \mathbb{E}_x \int_0^{\hat{\tau}_{(a,y)}} e^{-rt} (\hat{\mathcal{G}}_r \psi)(\hat{X}_t) dt,$$

where $\hat{\tau}_{(a,y)} = \inf\{t \geq 0 : \hat{X}_t \geq y\}$. Since $\hat{X}_{\hat{\tau}_{(a,y)}} = y$ a.s. and $(\hat{\mathcal{G}}_r \psi)(x) = \left((\hat{\mathcal{G}}_r - \mathcal{G}_r + \mathcal{G}_r)\psi\right)(x) = \left((\hat{\mathcal{G}}_r - \mathcal{G}_r)\psi\right)(x) = \frac{1}{2}(\hat{\sigma}^2(x) - \sigma^2(x))\psi''(x) \leq 0$ by the X-harmonicity and the convexity of $\psi(x)$, we find that

$$\mathbb{E}_x[e^{-r\hat{\tau}_{(a,y)}}]\psi(y) = \frac{\hat{\psi}(x)}{\hat{\psi}(y)}\psi(y) \leq \psi(x)$$

from which the alleged results follow by the nonnegativity of the payoff $g(x)$. Establishing the reverse conclusions in case the fundamental solution $\psi(x)$ is concave is completely analogous. □

Theorem 5.1 extends previous findings based on continuous diffusions to the present setting as well and states a set of conditions in terms of the convexity (concavity) of the fundamental solution $\psi(x)$ under which increased volatility unambiguously decelerates (accelerates) rational exercise by expanding (shrinking) the continuation region where waiting is optimal. As is clear from this observation, the sign of the relationship between increased volatility and the optimal stopping policy is a process-specific property that as such does not depend on the precise form of the exercise payoff as long as the supremum at which the expected present value of the payoff is maximized exists and constitutes the optimal stopping rule.

It is worth noticing that the proof of our Theorem 5.1 indicates that the analysis of the impact of increased volatility on the optimal policy and its value reduces to the comparison of the r-superharmonic mappings characterized by the integro-differential operators $\mathcal{G}_r$ and $\hat{\mathcal{G}}_r$. Since $(\hat{\mathcal{G}}_r u)(x) \leq (\mathcal{G}_r u)(x)$ for any sufficiently smooth convex function $u : I \mapsto \mathbb{R}_+$ and $(\hat{\mathcal{G}}_r v)(x) \geq (\mathcal{G}_r v)(x)$ for any sufficiently smooth concave function $v : I \mapsto \mathbb{R}_+$, we find that the findings of our Theorem 5.1 generate a natural ordering for the convex (concave) solutions of the variational inequalities $\max\{(\hat{\mathcal{G}}_r u)(x), g(x) - u(x)\} = 0$ and $\max\{(\mathcal{G}_r u)(x), g(x) - u(x)\} = 0$.

We state next sufficient conditions for convexity of the value when the underlying process is the slightly less general

$$X_t = \int_0^t \mu(X_s)ds + \int_0^t \sigma X_s dW_s + \int_0^t \int_{\mathcal{S}(m)} \gamma(z) X_s \tilde{N}(dz, ds), \tag{19}$$

where the diffusion and jump components are assumed to be linear in the state variable. In this setting we can state sufficient conditions for the convexity of the value function.

Theorem 5.2. *Suppose that g and μ are convex functions, and that*

$$rx - \mu(x)$$

is increasing. Then the value function of the stopping problem is convex.

Proof. We denote $Y_t^1 := \frac{\partial X_t}{\partial x}$. By virtue of Theorem V.40 of Protter (2004), we can differentiate the flow $X_t = X_t^x$ with respect to the initial state x to obtain

$$Y_t^1 = \int_0^t \mu'(X_s^x)Y_s^1 ds + \int_0^t \sigma Y_s^1 dW_s + \int_0^t \int_{\mathcal{S}(m)} \gamma(z) Y_s^1 \tilde{N}(dz, ds),$$

which implies that

$$Y_t^1 = \exp\left(\int_0^t \mu'(X_s^x)ds\right) M_t \geq 0,$$

where

$$M_t = \exp\left(\sigma \int_0^t dW_s - \frac{1}{2}\sigma^2 t + \int_0^t \int_{\mathcal{S}(m)} \ln(1+\gamma(z))N(ds, dz) - \lambda\overline{\gamma}t\right)$$

is an exponential martingale independent of x and

$$\overline{\gamma} := \int_{\mathcal{S}(m)} \gamma(z)m(dz).$$

Thus differentiating the mapping

$$Q(t,x) := \mathbb{E}\left[e^{-rt}g(X_t^x)\right]$$

with respect to x yields

$$Q_x(t,x) = \mathbb{E}\left[\exp\left(-\int_0^t (r-\mu'(X_s^x))ds\right) g'(X_t^x)M_t\right] \geq 0,$$

which as a function of x is increasing, being under our assumptions the product of two non-negative and monotonically increasing functions. Thus $Q(t,x)$ is an increasing and convex function of x. Consequently, all elements of the increasing sequence $\{V_k(x)\}_{k\in\mathbb{N}}$ defined by

$$V_0(x) = \sup_{t\geq 0} \mathbb{E}\left[e^{-rt}g(X_t^x)\right]$$
$$V_{k+1}(x) = \sup_{t\geq 0} \mathbb{E}\left[e^{-rt}V_k(X_t^x)\right]$$

are increasing and convex. Furthermore, $V_k(x) \uparrow V(x)$. If $\alpha \in [0,1]$ and $x, y \in I$, then

$$\begin{aligned} \alpha V(x) + (1-\alpha)V(y) &\geq \alpha V_k(x) + (1-\alpha)V_k(y) \\ &\geq V_k(\alpha x + (1-\alpha)y) \end{aligned}$$

for all k. By monotone convergence

$$\alpha V(x) + (1-\alpha)V(y) \geq \lim_{k\to\infty} V_k(\alpha x + (1-\alpha)y) = V(\alpha x + (1-\alpha)y),$$

which implies the convexity of the value V. □

Theorem 5.2 states a set of conditions under which the sign of the relationship between increased volatility and the value of the considered optimal stopping problem is unambiguously positive. It is worth noticing that along the lines of the findings by Alvarez (2003) the monotonicity of the net appreciation rate $\mu(x) - rx$ is the key factor determining how higher volatility affects the optimal policy. The reason for this observation is naturally the fact that our evaluations are based on the compensated compound Poisson process (which is a martingale). If this were not the case, then the local expected behavior of the underlying jump process would naturally have a constant effect on the monotonicity requirement stated in Theorem 5.2.

Having characterized the impact of increased volatility on the optimal policy and its value, it is naturally of interest to analyze how the jump-intensity λ measuring the rate at which the downside risk is realized affects these factors. Along the lines of our previous notation, we now consider two jump diffusions of the form (4), X and $\hat{X}$, which are otherwise identical but are subject to different jump intensities, $\lambda > \hat{\lambda}$. In line with this notation, we denote the values of the associated optimal stopping problems again by V_λ and $V_{\hat{\lambda}}$, the associated differential operators by $\mathcal{G}_r$ and $\hat{\mathcal{G}}_r$, and the associated increasing fundamental solutions (given that assumption A1 is satisfied) by ψ_λ and $\psi_{\hat{\lambda}}$, respectively. Our main characterization on the impact of increased jump intensity on the value and the optimal policy is now summarized in our next theorem.

Theorem 5.3. *Assume that the increasing fundamental solution $\psi_\lambda(x)$ is convex. Then*

$$\frac{\psi_{\hat{\lambda}}(x)}{\psi_{\hat{\lambda}}(y)} \leq \frac{\psi_\lambda(x)}{\psi_\lambda(y)}$$

for all $x \leq y$. Hence,

$$\psi_{\hat{\lambda}}(x) \sup_{y \geq x} \left[\frac{g(y)}{\psi_{\hat{\lambda}}(y)} \right] \leq \psi_\lambda(x) \sup_{y \geq x} \left[\frac{g(y)}{\psi_\lambda(y)} \right]$$

provided that the supremum exists. Moreover, if the conditions of Theorem 3.3 are satisfied, then $V_\lambda(x) \geq V_{\hat{\lambda}}(x)$ and, therefore,

$$C_{\hat{\lambda}} = \{x \in I : V_{\hat{\lambda}}(x) > g(x)\} \subseteq \{x \in I : V_\lambda(x) > g(x)\} = C_\lambda.$$

If the increasing fundamental solution $\psi_{\hat{\lambda}}(x)$ is concave, then the inequalities and inclusions stated above are reversed.

Proof. The assumed convexity of the increasing fundamental solution $\psi_\lambda(x)$ implies that $\psi_\lambda(x+\gamma(x,z)) \geq \psi_\lambda(x) + \psi'_\lambda(x)\gamma(x,z)$ for any $z \in \mathcal{S}(m)$ and, therefore, that

$$\int_{\mathcal{S}(m)} \{\psi_\lambda(x+\gamma(x,z)) - \psi_\lambda(x) - \psi'_\lambda(x)\gamma(x,z)\} m(dz) > 0.$$

Consequently, we observe that

$$(\hat{\mathcal{G}}_r\psi_\lambda)(x) = (\hat{\lambda}-\lambda)\int_{\mathcal{S}(m)} \{\psi_\lambda(x+\gamma(x,z)) - \psi_\lambda(x) - \psi'_\lambda(x)\gamma(x,z)\} m(dz) < 0$$

for all $x \in I$. Applying now Dynkin's theorem to $\psi_\lambda(x)$ then finally proves that $\psi_{\hat{\lambda}}(x)/\psi_{\hat{\lambda}}(y) \leq \psi_\lambda(x)/\psi_\lambda(y)$ for $x \leq y$. The rest of the alleged results then follow from the nonnegativity of the payoff $g(x)$ and Theorem 3.3. Establishing the reverse conclusions in the case where $\psi_{\hat{\lambda}}(x)$ is concave is completely analogous. □

Theorem 5.3 characterizes how the direction of the impact of increased jump-intensity λ on the optimal stopping policy and its value can be unambiguously determined when the fundamental solution is convex (concave). Along the lines of our findings on the impact of increased volatility, we observe that higher jump-intensity also slows down (speeds up) rational exercise by expanding (shrinking) the continuation region when $\psi(x)$ is convex (concave). This result is economically important, since it essentially states that if the value is convex on the continuation region where exercising is suboptimal, then the combined impact of downside risk and systematic market risk on the exercise incentives of rational investors is unambiguously negative.

6 Explicit Illustrations

In this section our objective is to illustrate our main findings within explicitly parametrized examples based on different descriptions for the underlying stochastic dynamics. As usually, we first illustrate our findings for the arithmetic

Lévy process and the geometric Lévy process since in those cases the representation obtained in the analysis of our previous sections is valid. We then extend our findings to cover also two other solvable cases: namely the constant elasticity of variance case and a logistic jump-diffusion case.

6.1 Arithmetic Stochastic Dynamics

In the arithmetic case

$$dL_t = \mu dt + \sigma dW_t + \int_{(0,\infty)} \gamma z \tilde{N}(dt, dz)$$

(with $\gamma < 0$ so that *X2* holds) $I = \mathbb{R}$ and a sufficient condition for assumption *X1* to hold is $\mu + \gamma\lambda\overline{m} > 0$. The associated integro-differential equation

$$\frac{1}{2}\sigma^2\psi''(x) + (\mu + \gamma\lambda\overline{m})\psi'(x) - (r + \lambda)\psi(x) + \lambda \int_{(0,\infty)} \psi(x + \gamma z)m(dz) = 0 \quad (20)$$

has an increasing solution $e^{k_1 x}$ where $k_1 > 0$ solves

$$\frac{1}{2}\sigma^2 k^2 + (\mu + \gamma\lambda\overline{m})k + \lambda \int_{(0,\infty)} e^{\gamma z k} m(dz) - (r + \lambda) = 0. \quad (21)$$

In light of our general observations we find that for any reward function g satisfying conditions *g1* and *Ag2*, and such that $e^{-k_1 x} g(x)$ has a unique maximizer $x^* \in \mathbb{R}$ and is non-increasing for $x > x^*$, the value of the optimal stopping policy can be represented as

$$V(x) = e^{k_1 x} \sup_{y \geq x} \{e^{-k_1 y} g(y)\} = \begin{cases} g(x), & x \geq x^* \\ g(x^*)e^{k_1(x - x^*)}, & x < x^*, \end{cases} \quad (22)$$

where x^* is the unique maximizer of g/ψ, i.e. for a differentiable g the solution of $D_x[\ln g(x)] = k_1$. It is also worth pointing out that in accordance with the findings of our Theorem 4.1 we now find that the root $k_1 \in (\tilde{k}_1, \hat{k}_1)$, where

$$\hat{k}_1 = -\frac{\mu + \gamma\lambda\bar{m}}{\sigma^2} + \sqrt{\left(\frac{\mu + \gamma\lambda\bar{m}}{\sigma^2}\right)^2 + \frac{2(r + \lambda)}{\sigma^2}}$$

denotes the positive root of the characteristic equation $\sigma^2 k^2 + 2(\mu + \gamma\lambda\overline{m})k = 2(r + \lambda)$, and

$$\tilde{k}_1 = -\frac{\mu + \gamma\lambda\bar{m}}{\sigma^2} + \sqrt{\left(\frac{\mu + \gamma\lambda\bar{m}}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

denotes the positive root of the characteristic equation $\sigma^2 k^2 + 2(\mu + \gamma\lambda\overline{m})k = 2r$. Consequently, we observe that in the present setting

$$e^{\tilde{k}_1 x} \sup_{y \geq x} \left\{ e^{-\tilde{k}_1 y} g(y) \right\} \leq e^{k_1 x} \sup_{y \geq x} \left\{ e^{-k_1 y} g(y) \right\} \leq e^{\hat{k}_1 x} \sup_{y \geq x} \left\{ e^{-\hat{k}_1 y} g(y) \right\}$$

provided that the maximum exists. Moreover, given that in the present case $\tilde{\psi}_\theta(x) = e^{K_\theta x}$, where

$$K_\theta = -\frac{\mu + \gamma\lambda\bar{m}}{\sigma^2} + \sqrt{\left(\frac{\mu + \gamma\lambda\bar{m}}{\sigma^2}\right)^2 + \frac{2\theta}{\sigma^2}}$$

we observe that $\operatorname{argmax}\{e^{-k_1 y} g(y)\} = \operatorname{argmax}\{e^{-K_\theta y} g(y)\}$ whenever the identity

$$\theta = r + \lambda \int_{(0,\infty)} (1 - e^{\gamma z k}) m(dz) \tag{23}$$

holds. This observation is important since it demonstrates that in the present case both the value as well as the optimal stopping rule of the optimal stopping problem (3) of the underlying jump diffusion coincides with the value and stopping rule of the associated stopping problem of a continuous diffusion by properly adjusting the discount rate. Hence, our results indicate that whenever the value of the optimal policy admits the representation (22) the jump-risk can be viewed as a discount rate effect as characterized by the identity (23). More precisely, whenever the value of the optimal policy admits the representation (22) we have that $\tilde{V}_\theta(x) = V(x)$ by choosing the discount rate according to the identity (23).

It is worth noticing that according to our general results the strict convexity of the increasing fundamental solution $e^{k_1 x}$ implies that increased volatility σ as well as higher jump-intensity λ increases the value of the optimal stopping policy and raises the optimal boundary at which the underlying jump-diffusion should be stopped. To see that this is indeed the case consider the mapping

$$\bar{P}(\lambda, \sigma, k) = (\mu + \gamma\lambda\overline{m})k + \frac{1}{2}\sigma^2 k^2 + \lambda \int_{(0,\infty)} e^{\gamma z k} m(dz) - (r + \lambda).$$

Standard differentiation yields that $\bar{P}_\sigma(\lambda, \sigma, k) = \sigma k^2 > 0$ and

$$\bar{P}_\lambda(\lambda, \sigma, k) = \int_{(0,\infty)} \{e^{\gamma z k} - 1 + \gamma k z\} m(dz) > 0.$$

Therefore, the inequality $\bar{P}(\lambda,\sigma,0) = -r < 0$, the limiting condition $\bar{P}(\lambda,\sigma,k) \uparrow +\infty$ as $k \to \infty$, and the strict convexity of the function $\bar{P}(\lambda,\sigma,k)$ imply that $\partial k_1/\partial\sigma < 0$ and $\partial k_1/\partial\lambda < 0$ and, therefore, that $\partial e^{k_1(x-y)}/\partial\sigma > 0$ and $\partial e^{k_1(x-y)}/\partial\lambda > 0$ for all $x \le y$.

As a numerical illustration, consider the *capped option* reward function

$$g(x) = \max\{0, p\min(K,x) - qK\},$$

where we assume $p > q > 0$ and $K \le \frac{p}{p-q}\frac{1}{k_1} =: K_0$ to guarantee that g/ψ is maximized at K (if $K > K_0$, the maximizer is an interior point of $(0,K)$, see Alvarez (1996) for a detailed analysis and interpretation in the continuous setting). As stated, in this case g/ψ attains a unique maximum value at $x^* = K$, which is a point of nondifferentiability for g. Assumption *g1* is now satisfied and if *Ag2* holds, the value of the optimal stopping problem is

$$V(x) = e^{k_1 x}\sup_{y\ge x}\left\{e^{-k_1 y}g(y)\right\} = \begin{cases}(p-q)K, & x \ge K\\ e^{k_1(x-K)}(p-q)K, & x < K\end{cases} \tag{24}$$

This is a continuous function, but its derivative has a discontinuity at x^*:

$$\lim_{x\to x^*-} V'(x) = k_1(p-q)K > 0 = \lim_{x\to x^*+} g'(x) = \lim_{x\to x^*+} V'(x),$$

and there is no smooth pasting. The graphs of the reward function, the function g/ψ and the value and its derivative for $p = 1$, $q = 0.5$, $r = 0.04$, $\mu = 0.075$, $\sigma = 0.1$, $\lambda = 0.1$, $\gamma = -0.5$, and $K = 0.75 \cdot K_0$ (implying that $k_1 = 1.1463$ and $K_0 = 1.7447$) are shown in Figure 1 (for these values it can be checked numerically that *Ag2* holds).

6.2 Geometric Stochastic Dynamics

Geometric processes have been of paramount importance in mathematical finance for several decades, with the most extensively used and well-known instance being the geometric Brownian motion $S_t = s_0\exp\{\mu t + \sigma W_t\}$, where $\sigma > 0$ and W is a standard Wiener process. During the last decade, a considerable amount of research has been done on geometric Lévy models

$$Y_t = y_0\exp\{\alpha t + \sigma W_t + J_t\}, \tag{25}$$

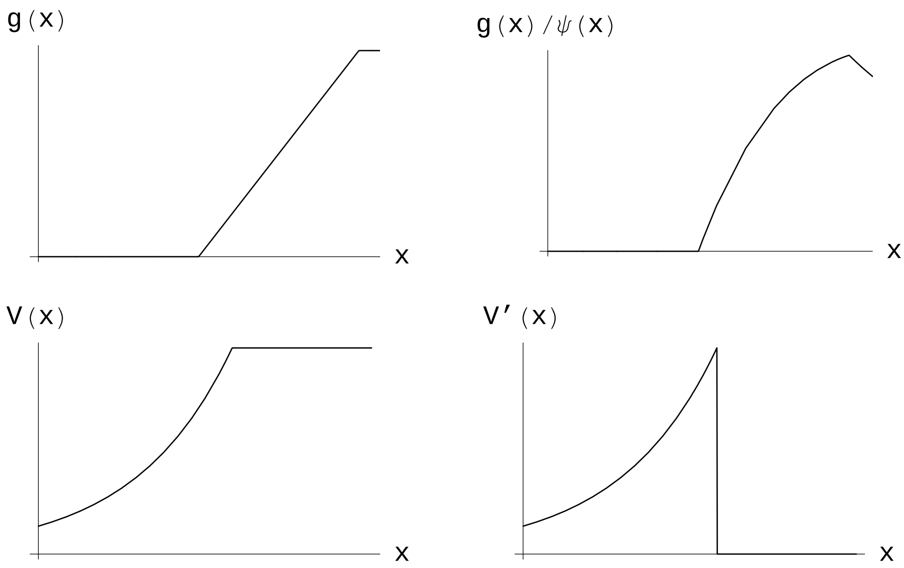

Figure 1: *The reward function g, the function g/ψ, the value function V and the derivative of the value V' for the capped option case*

where in addition to the deterministic drift and the Gaussian component there is a jump process J_t in the exponent.

A geometric Lévy process $Y = \{Y_t\}$ with a finite Lévy measure $\nu = \lambda m$ is a jump diffusion whose dynamics are given by

$$dY_t = Y_{t-}\left\{\alpha dt + \sigma dW_t + \lambda \int_{\mathcal{S}(m)} \gamma(z)(N(dt,dz) - \nu(dz)dt)\right\}, \tag{26}$$

where both the drift α and the diffusion coefficient σ are assumed to be positive. Note that in this case $I = \mathbb{R}_+$ and the explicit solution Y_t equals

$$y_0 \exp\Big\{\tilde{\alpha}t + \sigma W_t + \int_0^t \int_{\mathcal{S}(m)} \ln(1+\gamma(z))\tilde{N}(ds,dz)\Big\}. \tag{27}$$

where $\tilde{\alpha} = \alpha - \frac{1}{2}\sigma^2$. To ascertain that *X1* holds, we require that $\tilde{\alpha} > 0$. For simplicity of exposition, we take $\gamma(z) = -z$ and to guarantee that *X2* is satisfied, we assume $\mathcal{S}(m) \subseteq (0,1)$. Furthermore, to ensure the finiteness of the value of the optimal stopping problem, we need to impose the integrability condition $\alpha - r < 0$ (which is known in the literature on financial economics as the absence of speculative bubbles condition). The integro-differential operator $\mathcal{G}_r$ takes now the form

$$\frac{1}{2}\sigma^2 x^2 \psi''(x) + \hat{\alpha} x \psi'(x) - (r+\lambda)\psi(x) + \lambda \int_0^1 \psi(x - xz) m(dz) = 0, \tag{28}$$

where $\hat{\alpha} = \alpha + \lambda \bar{m}$. By guessing now the solution to be of form x^k, we obtain the characteristic equation for k:

$$\frac{1}{2}\sigma^2 k(k-1) + (\alpha + \lambda \bar{m})k - (r+\lambda) + \lambda \int_0^1 (1-z)^k m(dz) = 0, \tag{29}$$

If the integrability condition is satisfied, it is easy to show that this equation has a solution $k_1 > 1$, and thus $\psi(x) = x^{k_1}$ is an increasing smooth solution of (28) which vanishes at $x = 0$. Hence assumption *A1* is satisfied. In light of our representation of the value of the optimal policy in terms of an associated non-linear programming problem, we find that for any reward function g satisfying conditions *g1* and *Ag2*, and such that $x^{-k_1}g(x)$ has a unique maximizer $x^* \in \mathbb{R}$ and is non-increasing for $x > x^*$, the value of the optimal stopping policy can be represented as

$$V(x) = x^{k_1} \sup_{y \geq x} \left\{ y^{-k_1} g(y) \right\} = \begin{cases} g(x), & x \geq x^* \\ g(x^*)(x/x^*)^{k_1}, & x < x^*, \end{cases} \tag{30}$$

where x^* is the unique maximizer of g/ψ, i.e. for a differentiable g the solution of $g'(x^*)x^*/g(x^*) = k_1$.

As in the arithmetic case, we observe that our Theorem 4.1 implies that in the present case the root of the equation (29) k_1 satisfies the condition $k_1 \in (\tilde{k}_1, \hat{k}_1)$, where

$$\hat{k}_1 = \frac{1}{2} - \frac{\alpha + \lambda \bar{m}}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\alpha + \lambda \bar{m}}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}}$$

denotes the positive root of the characteristic equation $\sigma^2 k(k-1) + 2(\alpha+\lambda\bar{m})k - 2(r+\lambda) = 0$ and

$$\tilde{k}_1 = \frac{1}{2} - \frac{\alpha + \lambda \bar{m}}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\alpha + \lambda \bar{m}}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

denotes the positive root of the characteristic equation $\sigma^2 k(k-1) + 2(\alpha+\lambda\bar{m})k - 2r = 0$. To demonstrate this observation, consider the behavior of the function

$$P(\lambda, \sigma, k) = \frac{1}{2}\sigma^2 k(k-1) + (\alpha + \lambda \bar{m})k - (r+\lambda) + \lambda \int_0^1 (1-z)^k m(dz).$$

We first observe that

$$P(\lambda,\sigma,\hat{k}_1)=\lambda\int_0^1(1-z)^{\hat{k}_1}m(dz)>0$$

and

$$P(\lambda,\sigma,\tilde{k}_1)=\lambda\int_0^1((1-z)^{\tilde{k}_1}-1)m(dz)<0.$$

However, since $P(\lambda,\sigma,1)=\alpha-r<0$ and $P(\lambda,\sigma,k)$ is strictly convex on $k>1$ we observe that $\tilde{k}_1<k_1<\hat{k}_1$ and, therefore, that

$$x^{\hat{k}_1}\sup_{y\geq x}\left[g(y)y^{-\hat{k}_1}\right]\leq x^{k_1}\sup_{y\geq x}\left[g(y)y^{-k_1}\right]\leq x^{\tilde{k}_1}\sup_{y\geq x}\left[g(y)y^{-\tilde{k}_1}\right]$$

provided that the maximum exists. Moreover, since in this case $\tilde{\psi}_\theta(x)=x^{l_\theta}$, where

$$l_\theta=\frac{1}{2}-\frac{\alpha+\lambda\bar{m}}{\sigma^2}+\sqrt{\left(\frac{1}{2}-\frac{\alpha+\lambda\bar{m}}{\sigma^2}\right)^2+\frac{2\theta}{\sigma^2}}$$

we observe that $\operatorname{argmax}\{y^{-k_1}g(y)\}=\operatorname{argmax}\{y^{-l_\theta}g(y)\}$ provided that the identity

$$\theta=r+\lambda\int_0^1(1-(1-z)^k)m(dz) \tag{31}$$

is satisfied. Along the lines indicated by our observations in the arithmetic case, we again observe that the effect of jump-risk on valuation can be captured by making an appropriate adjustment in the discount rate of the stopping problem of the associated continuous diffusion as is characterized by (31). Hence, whenever the value of the optimal stopping problem (3) admits the representation (30) we have $\tilde{V}_\theta(x)=V(x)$ by choosing the discount rate according to the identity (31).

It is also clear from our analysis that the increasing fundamental solution is strictly convex in this case as well. Thus, as our results in Theorem 5.1 and in Theorem 5.3 indicated, increased volatility and higher jump-intensity should increase the value and decelerate exercise by increasing the optimal stopping boundary. To see that this is indeed the case in the present example, we first observe that $P_\sigma(\lambda,\sigma,k)=\sigma k(k-1)>0$ for $k>1$ and

$$P_\lambda(\lambda,\sigma,k)=\bar{m}k-1+\int_0^1(1-z)^k m(dz)=\mathbb{E}[zk-1+(1-z)^k].$$

Since the function $z \mapsto zk - 1 + (1-z)^k$ is strictly convex for $k > 1$ and attains a minimum at $z = 0$, we observe that $\mathbb{E}[zk - 1 + (1-z)^k] > 0$ and, therefore, that $P_\lambda(\lambda, \sigma, k) > 0$ for all $k > 1$. Since the positive root k_1 is attained on this set and $P(\lambda, \sigma, 1) = \alpha - r < 0$, we find that $\partial k_1/\partial\lambda < 0$ and $\partial k_1/\partial\sigma < 0$. An interesting implication of this observation is that

$$\frac{\partial}{\partial\sigma}\left(\frac{x}{y}\right)^{k_1} > 0 \quad \text{and} \quad \frac{\partial}{\partial\lambda}\left(\frac{x}{y}\right)^{k_1} > 0$$

for all $x \leq y$. Consequently, we observe that both increased volatility as well as higher jump-intensity increases the value of the problem and postpones exercise by raising the threshold at which the underlying process should be optimally stopped. Moreover, if the exercise payoff is continuously differentiable at the exercise boundary x^*, then

$$\frac{\partial}{\partial\sigma}\left[\frac{g'(x^*)x^*}{g(x^*)}\right] = \frac{\partial k_1}{\partial\sigma} < 0 \quad \text{and} \quad \frac{\partial}{\partial\lambda}\left[\frac{g'(x^*)x^*}{g(x^*)}\right] = \frac{\partial k_1}{\partial\lambda} < 0.$$

In other words, both increased volatility and higher jump intensity decreases the elasticity of the exercise payoff at the optimal exercise threshold x^*.

For the sake of explicit illustration, we now consider the case of a linear reward function in the geometric Lévy model. Let $g(x) = \max(ax - b, 0)$ with $a, b > 0$. This case contains the standard American call option (take $a = 1$, $b = K$), and also the rewards of optimal stopping problems associated with irreversible investment decisions (see Boyarchenko (2004) for a very readable account on the relationship between perpetual American options and irreversible investment decisions). Clearly, the increasing function g satisfies *g1*. The function g/ψ has a unique maximum in $\mathbb{R}_+$ and is non-increasing for argument values larger than the maximizer, since the sign of

$$D_x\left[\frac{g(x)}{\psi(x)}\right] = \frac{x^{k_1-1}(ax - (ax-b)k_1)}{x^{2k_1}}$$

depends only on the linear decreasing function $a(1-k_1)x + k_1 b$, so *Ag1* is satisfied. If *Ag2* is satisfied, then by theorem 3.3, the value of the optimal stopping problem can now be represented as

$$V(x) = x^{k_1}\sup_{y \geq x}\left\{y^{-k_1}(ay - b)\right\} = \begin{cases} ax - b, & x \geq x^* \\ (ax^* - b)(x/x^*)^{k_1}, & x < x^* \end{cases} \tag{32}$$

where $x^* = \frac{k_1 b}{(k_1-1)a}$ is the unique maximizer of the function $g(x)/\psi(x)$. As usually in the real options literature on irreversible investment, we notice that the option multiplier $M = k_1/(k_1-1)$ determines the comparative static properties of the optimal exercise threshold x^*. In light of our findings this multiplier reads $\tilde{M} = \tilde{k}_1/(\tilde{k}_1-1)$ and $M = \hat{k}_1/(\hat{k}_1-1)$ for the stopping problems of the associated continuous diffusion. We illustrate these option multipliers in Figure 2 for Beta(c,d)-distributed jumps under the assumption that $\alpha = 0.02, r = 0.035, \lambda = 0.01, a = b = 1, c = 1.25, d = 2$. As Figure 2 indicates, the option multipliers

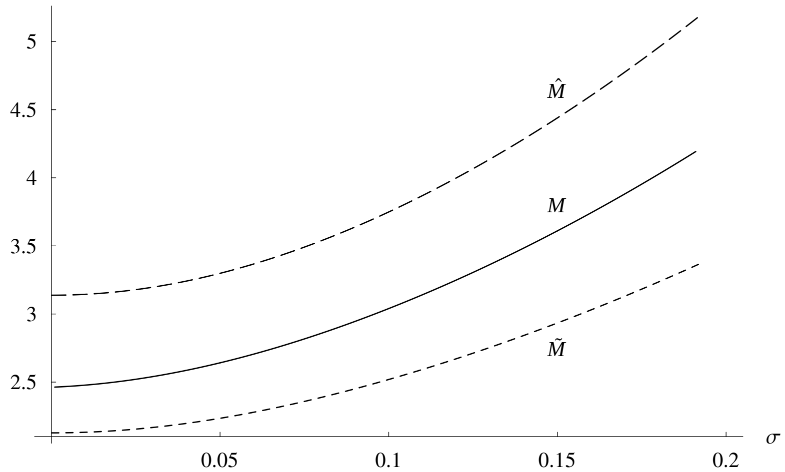

Figure 2: The impact of volatility on the option multipliers M, $\tilde{M}$, and $\hat{M}$

are increasing as functions of the underlying volatility coefficient. Moreover, the option multipliers satisfies the condition $M \in (\tilde{M}, \hat{M})$ as was established in our Theorem 4.1. The values of the optimal stopping problems are graphically illustrated for Beta(c,d)-distributed jumps in Figure 3 under the assumption that $\alpha = 0.02, r = 0.035, \lambda = 0.01, a = b = 1, c = 1.25, d = 2, \sigma = 0.1$ (which implies that $x^* = M = 2.95, \hat{x}^* = \hat{M} = 3.75$, and $\tilde{x}^* = \tilde{M} = 2.52$) Figure 3 illustrates explicitly the results of our Theorem 4.1 for the values of the stopping problems. It is of interest to notice that as was predicted by Theorem 4.1, the value $V(x)$ of the considered stopping problem is sandwiched between the two values $\tilde{V}_{r+\lambda}(x)$ and $\tilde{V}_r(x)$.

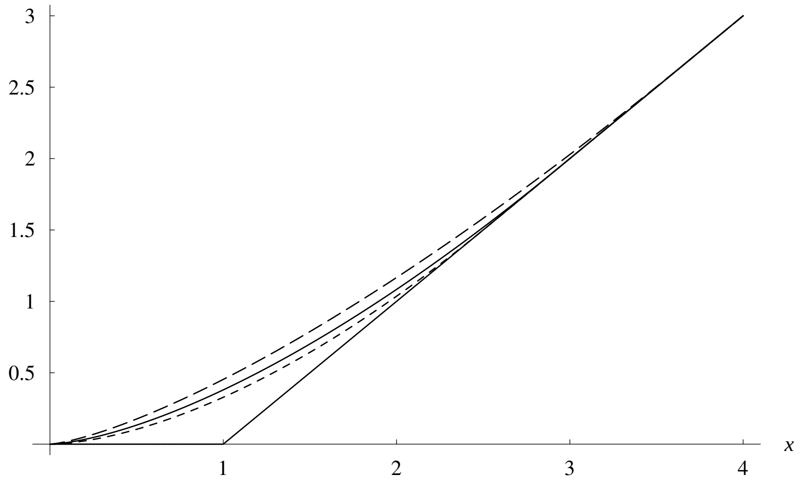

Figure 3: The exercise payoff $(x-1)^+$ and the values $V(x), \tilde{V}_r(x)$, and $\tilde{V}_{r+\lambda}(x)$

Consider next the concave reward function $g(x) = \max(\ln x, 0)$. Then function g satisfies *g1* and g/ψ has a unique maximum in $\mathbb{R}_+$ and is non-increasing for argument values larger than the maximizer, since the sign of

$$D_x\left[\frac{g(x)}{\psi(x)}\right] = \frac{1-k_1\ln x}{x^{k_1+1}}$$

depends only on the decreasing function $1-k_1\ln x$, so *Ag1* is satisfied. If *Ag2* is satisfied, by theorem 3.3, the value of the optimal stopping problem can now be represented as

$$V(x) = x^{k_1}\sup_{y\geq x}\left\{y^{-k_1}\ln y\right\} = \begin{cases} \ln x, & x \geq \exp\{1/k_1\} \\ x^{k_1}\left[\frac{1}{ek_1}\right], & x < \exp\{1/k_1\} \end{cases} \tag{33}$$

where $x^* = \exp\{1/k_1\}$ is the unique maximizer of the function $g(x)/\psi(x)$. As was predicted by our Theorem 5.1 we find that under the assumption $r > \alpha$ we have $\partial x^*/\partial\sigma = -(x^*/k_1^2)\partial k_1/\partial\sigma > 0$ and $\partial x^*/\partial\lambda = -(x^*/k_1^2)\partial k_1/\partial\lambda > 0$. Hence, both increased volatility as well as higher jump-intensity decelerate optimal exercise by raising the optimal exercise boundary in this case as well. It is, however, worth noticing that in the present example the maximizing threshold x^* exists even when $k_1 < 1$, that is, even when the fundamental solution is not convex as a function of the state. Hence, for the exercise payoff $g(x) =$

max(ln x, 0) the condition $r > \alpha$ can be relaxed. If $0 < r \leq \alpha$ then $k_1 \in (0, 1]$ since in that case $P(\lambda, \sigma, 0) = -r < 0$ and $P(\lambda, \sigma, 1) = \alpha - r \geq 0$. Under those circumstances the sign of the relationship between increased volatility and the optimal exercise strategy is reversed as the root k_1 becomes an increasing function of volatility. More precisely, if $r \leq \alpha$ then $\partial x^*/\partial\sigma = -(x^*/k_1^2)\partial k_1/\partial\sigma < 0$ and $\partial x^*/\partial\lambda = -(x^*/k_1^2)\partial k_1/\partial\lambda < 0$. We illustrate this observation graphically for Beta(c, d)-distributed jumps in Figure 4 under the assumption that $\alpha = 0.04, r = 0.02, \lambda = 0.01, a = b = 1, c = 1.25$, and $d = 2$. Figure 4 illustrates

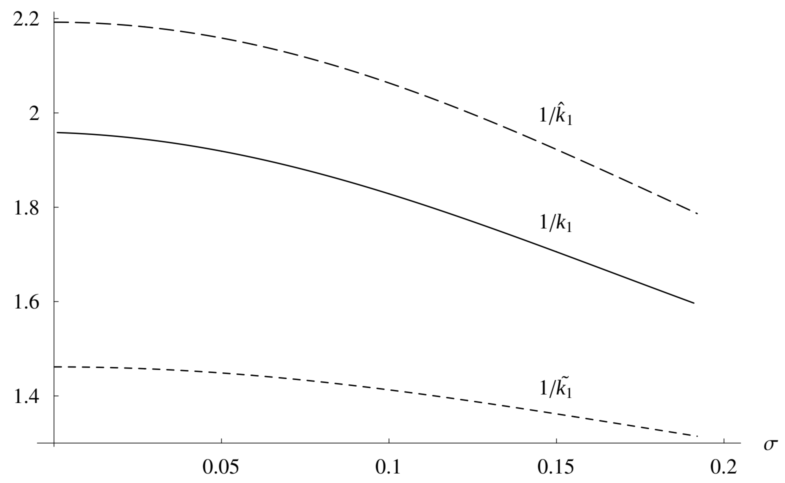

Figure 4: The impact of volatility on the exercise thresholds $1/k_1$, $1/\tilde{k}_1$, and $1/\hat{k}_1$

how the sign of the relationship between increased volatility and the optimal exercise threshold is reversed as the increasing fundamental solution becomes concave. It is worth noticing that even in this case the order of the exercise thresholds remain naturally unchanged since the ordering of the values $V(x)$, $\tilde{V}_{r+\lambda}(x)$, and $\tilde{V}_r(x)$ is based only on nonnegativity and monotonicity.

6.3 Constant Elasticity of Variance

Consider the *Constant Elasticity of Variance (CEV)* model with an added jump component

$$dS_t = S_t\Big\{rdt + \sigma S_t^{-\alpha} dW_t - \int_0^1 z\tilde{N}(dz, dt)\Big\}, \tag{34}$$

where $\alpha \in (0, 1)$ is a known exogenously determined constant, and suppose that the first exit time $\tau_{(0,x)} < \infty$ for any $x \in \mathbb{R}_+$. Note that the deterministic drift is now set equal to the discount rate r. We assume the reward to be of logarithmic utility type: $g(x) = \max(\ln x, 0)$. The associated integro-differential equation is now

$$\frac{1}{2}\sigma^2 x^{2(1-\alpha)}\psi''(x) + (r + \lambda\overline{m})x\psi'(x) + \lambda\int_0^1 \psi(x - xz)m(dz) = (r + \lambda)\psi(x), \tag{35}$$

which has an increasing solution $\psi(x) = x$, as can easily be verified. Now assumptions *X1*, *X2* and *A1* are clearly satisfied. Since the reward is of log utility type, we also have for $x > 1$

$$D_x[g(x)/\psi(x)] = \frac{1 - \ln x}{x^2},$$

whose sign depends on the decreasing function $1 - \ln x$; the unique maximizer is $x^* = e$ and assumption *Ag1* is satisfied. If *Ag2* holds, by theorem 3.3, the value of the optimal stopping problem has the representation

$$V(x) = x \sup_{y \geq x}\left\{\frac{\ln y}{y}\right\} = \begin{cases} \ln x, & x \geq e \\ \frac{x}{e} & x < e \end{cases} \tag{36}$$

as the unique maximum of $\ln x/x$ is $x = e$. It is worth noting that in this case we could not have taken a linear $g(x)$ as then *Ag1* is no longer satisfied. Due to our specific choice of parameters, the optimal value and the threshold are independent of the parameter values. For the associated diffusion, the increasing fundamental solution $\tilde{\psi}_\theta(x)$ of the characteristic differential equation can be expressed as

$$\tilde{\psi}_\theta(x) = xF_1\left(\frac{1}{2\alpha}\left(1 - \frac{\theta}{r + \lambda\overline{m}}\right), 1 + \frac{1}{2\alpha}, \frac{-r - \lambda\overline{m}}{\alpha\sigma^2}x^{2\alpha}\right),$$

where $F_1(a,b,z)$ is the Kummer confluent hypergeometric function. Notice that the solution depends on the values of parameters. As a numerical illustration, consider the case $(r, \lambda, \overline{m}, \sigma, \alpha) = (0.035, 0.1, 0.2, 0.1, 0.75)$. In this case $\tilde{x}^* \approx 1.54$ and $\hat{x}^* \approx 4.79$.

6.4 Logistic Jump Diffusion

In order to illustrate how Theorem 3.4 can be applied in the analysis of the stopping problem consider a stochastic process X such that the ratio $X/(1-X)$ evolves as a geometric Lévy process, i.e.

$$G_t := \frac{X_t}{1-X_t} = \frac{x}{1-x} \exp\big\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t + \int_0^t \int_{(0,1)} z\tilde{N}(ds, dz)\big\}. \quad (37)$$

Note that X lives on $I = (0,1)$. Being a C^2 function of a jump diffusion, $X_t = f(G_t) = G_t/(1+G_t)$ is also a jump diffusion and application of Itô formula yields the dynamics of X:

$$\begin{aligned} &dX_t = (1-X_t)X_t\{\mu + \lambda\overline{m} - X_t\sigma^2 + \tfrac{C(X_t)}{1-X_t}\}dt + \sigma(1-X_t)X_t dW_t + \\ &+X_t \textstyle\int_{(0,1)} \big\{\tfrac{1-z}{1-zX_t} - 1\big\}\tilde{N}(dt, dz), \end{aligned} \quad (38)$$

where $C(x) = \int_{(0,1)}\{\frac{1-z}{1-zx} - 1\}\nu(dz)$ arises from compensating the driving jump process in (38). Thus the integro-differential equation associated with the optimal stopping problem for X is

$$(1-x)x\alpha(x)\psi'(x) + \frac{1}{2}\sigma^2(1-x)^2x^2\psi''(x) + \int_{(0,1)} \psi(x + xc(x))\nu(dz) = \tilde{r}\psi(x), \quad (39)$$

where $\alpha(x) = \mu + \lambda\overline{m} - x\sigma^2$ and $c(x) = \frac{1-z}{1-zx} - 1$. Via transformation $\tilde{\psi}(y) := \psi(\frac{y}{y+1})$ equation (39) transforms into the following integro-differential equation:

$$\frac{1}{2}\sigma^2y^2\tilde{\psi}''(y) + (\mu + \lambda\overline{m})\tilde{\psi}'(y) + \int_{(0,1)} \tilde{\psi}(y - yz)\nu(dz) = \tilde{r}\tilde{\psi}(y), \quad (40)$$

which has an increasing solution $\tilde{\psi}(y) = y^{k_1}$, where k_1 is the positive root of equation

$$\frac{1}{2}\sigma^2k(k-1) + (\mu + \lambda\overline{m})k + \int_{(0,1)} (1-z)^k\nu(dz) = \tilde{r}.$$

Then by theorem 3.4 $\phi(x) := (\frac{x}{1-x})^{k_1}$ is an increasing solution of equation (39), and furthermore $\phi(0) = 0$. Since the process X by virtue of theorem

3.4 satisfies assumptions *X1* and *X2*, theorem 3.3 implies that for any reward function g satisfying assumptions *g1* and *Ag1–Ag2*

$$V(x) = \left(\frac{x}{1-x}\right)^{k_1} \sup_{y \geq x} \left\{ \frac{g(y)(1-y)^{k_1}}{y^{k_1}} \right\} = \begin{cases} g(x), & x \geq x^* \\ g(x^*)\left(\frac{x(1-x^*)}{x^*(1-x)}\right)^{k_1} & x < x^* \end{cases} \tag{41}$$

where x^* again satisfies the logarithmic derivative condition.

As a numerical illustration, consider a *bull spread* type reward (where $0 < K_1 < K_2 < 1$)

$$g(x) = \max\{0, x - K_1\} - \max\{0, x - K_2\}$$

in a logistic jump diffusion model with parameters

$$(\mu, \sigma, \gamma, \lambda, r, a, b, c, d, K_1, K_2) = (0.1, 0.3, -1, 0.1, 0.15, 1, 1, 1.5, 1, 0.4, 0.6)$$

and Beta(c, d) distributed jumps ($m(dz) = (1/\beta(c,d))z^{c-1}(1-z)^{d-1}dz$). These values lead to $k_1 = 1.2591$, and an increasing solution of the characteristic equation for the logistic process is given by $\psi(x) = \left(\frac{x}{1-x}\right)^{k_1}$. Thus

$$\frac{g(x)}{\psi(x)} = \begin{cases} \left(\frac{1-x}{x}\right)^{k_1}(K_2 - K_1), & x > K_2 \\ \left(\frac{1-x}{x}\right)^{k_1}(x - K_1), & K_1 < x \leq K_2 \\ 0, & x \leq K_1, \end{cases}$$

and this is decreasing for $x > K_2$ and has a unique maximum at

$$x^* = \frac{1-k_1}{2} + \frac{1}{2}\sqrt{(1-k_1)^2 + 4k_1K_1} > 0,$$

provided that $K_1 \leq x^* \leq K_2$ (in the case $x^* > K_2$ it is optimal to stop at K_2 and the value will not exhibit smooth pasting; for $x^* < K_1$, the value will be identically zero). By Theorems 3.3 and 3.4, for $K_1 \leq x^* \leq K_2$ the value function is given by

$$V(x) = \begin{cases} \max\{0, x - K_1\} - \max\{0, x - K_2\}, & x \geq x^* \\ (x^* - K_1)\left(\frac{x(1-x^*)}{x^*(1-x)}\right)^{k_1}, & x < x^*, \end{cases}$$

provided that *Ag1* and *Ag2* are satisfied. For the given parameters, *Ag1* holds, we have $x^* = 0.59185$ and we can establish numerically that *Ag2* holds. The value function is presented graphically in Figure 5.

To illustrate Theorem 5.1, consider the process $\hat{X}$ with parameters identical to those of X except that the volatility coefficient $\hat{\sigma} = 0.4 > 0.3 = \sigma$. We can compute $\hat{k}_1 = 1.21464$ and $\hat{x}^* = 0.59793 > 0.59185 = x^*$. This is in line with the theorem, and the value $\hat{V}(x)$ is depicted in Figure 5.

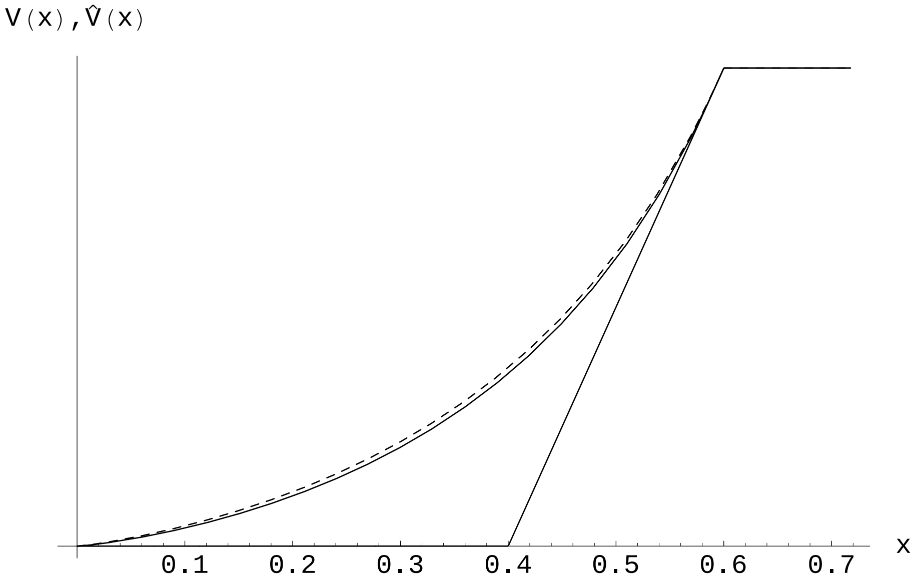

Figure 5: The impact of volatility on the value

7 Conclusions

In this study we generalized a representation result known to hold for continuous linear diffusions to include a class of spectrally one-sided Lévy diffusions: given some conditions, the optimal stopping problem for a one-dimensional spectrally negative Lévy diffusion can be reduced to an ordinary nonlinear programming problem. As the proof of our representation relied on the viscosity solution approach, differentiability is not required, and we are able to deal with nonsmooth reward functions as well. The class of processes for which the representation holds, contains the standard arithmetic and geometric Lévy processes. We established that the class is closed with respect to strictly increasing C^2 transforms, although the transform naturally changes the set of allowable reward functions.

Considering the fact that optimal stopping problems feature prominently in pricing of American options and in real options theory, reducing the stopping problem of a Lévy diffusion into a standard programming problem can significantly facilitate the ongoing research on these areas of mathematical finance. We demonstrated this by deriving several interesting comparative static properties of spectrally negative Lévy diffusions using our representation, and found out that a useful tool in obtaining bounds for the value of the optimal stopping of a Lévy diffusion is the corresponding stopping problem for an associated continuous diffusion. By choosing the discount rates appropriately (namely, as r and $r+\lambda$), we were able to sandwich the value of the jump diffusion problem between the values of two optimal stopping problems of this continuous diffusion. In fact, our findings indicate the existence of a critical discount rate θ^* such that the value and the threshold of the stopping problem of the jump diffusion with discount rate r coincide with the value and the threshold for the stopping problem of the associated diffusion with discount rate θ^*. Furthermore, it turned out that the impact of volatility on the optimal policy and its value in our setting is similar to the continuous case: for values convex (concave) below the optimal threshold, increased risk decelerates (accelerates) rational investment by expanding or leaving unchanged (shrinking or leaving unchanged) the continuation region and increasing or leaving unchanged (decreasing or leaving unchanged) the optimal threshold and the value of waiting. The impact of downside risk as measured by the intensity of the compound Poisson jump process on the optimal value was found out to be similar to the impact of the diffusion risk (as measured by the volatility). We also established that the key factor determining the relevant convexity/concavity properties of the value is (provided that it exists) the increasing fundamental solution of the associated integro-differential equation, which is process-specific. Thus we saw that the impact of volatility or downside risk is not dependent on the precise form of the exercise payoff, as long as the conditions for the optimality of the stopping rule characterized by a single threshold are met.

Motivated by our views on the importance of taking into account the downside risk, we concentrated our attention on the spectrally negative case with an increasing reward. However, the corresponding results can (with obvious modifications) be shown to hold for spectrally positive Lévy diffusions and decreasing reward functions.

In addition to their usefulness in obtaining information about the comparative static properties of Lévy diffusions and their relations (similarities and differences) to the continuous diffusion case, our results raise a few interesting questions. Firstly, it would be of interest to obtain precise knowledge on the scope of applicability of our representation. This boils largely down to the question: when is the assumption on the existence of an increasing smooth solution to the characteristic integro-differential equation true, and can conveniently verifiable sufficient conditions for this be found? Secondly, could a more convenient (i.e. analytically verifiable) substitute for our condition *Ag2* be derived, and if so, how generally applicable this substitute would be? The answers to these rather difficult questions, however, are outside the scope of the present study, and are therefore left for future research.

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