# NASH-IMPLEMENTATION WHEN THE PLANNER CAN FREELY USE REDUNDANT ALTERNATIVES

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October 20, 2008

**Abstract**

In the literature on Nash-implementation, it is not usually assumed that the range of social choice correspondences has to be full. This means that a given instance of Nash-implementation problem may involve social alternatives that are never actually implemented *i.e* not implemented in any environment. But these alternatives still matter, since they can be used in designing the game form. In this paper we shall show that every social choice correspondence can be Nash-implemented when the planner is able to freely use redundant alternatives. These are alternatives that don't belong to the range of choice correspondence and enter every agents every preference profile any way chosen. Of coure in practice these conditions are sometimes met and other times not. Still, this fact does turn the Nash-implementation problem even more heavily as one of social engineering.

**JEL Classification: C70, D71**

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[†] I have benefitted from discussions with professor Hannu Vartiainen, which have improved my thinking on implementation theory and mechanism design. All the omissions and errors are my own.

# 1 Introduction

The aim of this paper is to generalize and unify two ideas in the literature, one presented by Moore and Repullo (1990) and the other one by Mirrlees (1971). Moore and Repullo (1990) noted that in almost any social choice problem the planner can invent new alternative that is the worst for every agent. This naturally requires, that we interprete the range of social choice correspondence as the set of all relevant alternatives to make a decision, not all imaginable ones. This raises a first question: when all the decision relevant alternatives are given, how often is it in practice possible to invent new alternatives that enter at any position of all the preferences involved? Chapter 2 is devoted to an example that clarifies this idea.

As already mentioned, the second idea that is central for this paper comes from Mirrlees (1971). In this article an optimal income tax schedule is made incentive compatible by assuming a so called *Spence-Mirrlees -condition*, also known as a *single crossing -condition.* This condition works by guaranteeing monotonicity, in a sense it restricts the preference domain so that monotonicity hold trivially. This raises a second question: could we somehow more generally trivialize both conditions that are sufficient for Nash-implementation[1], monotonicity and no veto power, to guarentee Nash-implementability?[2]

Taken together, these two questions raise a third one that will give a partial aswer to both and that we are going to investigate more deeply here. This can be stated as follows: if the social planner can freely use new alternatives, can monotonicity and no veto power always be trivialized, so that practically any choice correspondence becomes Nash-implementable? The aswer to this question seems to be (quite generally) in the affirmative. At outset, the

[1] See Maskin (1999).

[2] Notice that no veto power always holds in economic settings, so Spence-Mirrlees condition actually guarantees Nash-implementability.

assumption that planner can freely use new alternatives entering at any position of different preference profiles, may feel impractical. This is true, but we shall see that the use of new alternatives has to be allowed only in a very restricted sense to obtain the result.

The paper is organized as follwing. Chapter 2 presents basic notations and definitions. We shall need few new definitions and some old ones need to be generalized. Chapter 3 is devoted to an example that ties the otherwise theoretical results in reality. After some intuition has been created, we present and prove the main result in chapter 4. Here it is also explained why this abstract result can be expected to hold in practice. Chapter 5 then concludes.

## 2 Notation and Definitions

Let $N = \{1, \ldots, n\}$ be the set of agents. We shall denote a generic element of this set by $i$ or $j$. Furthermore, let $A$ denote the set of social alternatives and $\mathscr{R}_A$ the set of all possible preferences in $A$ (the unrestricted domain).[3] By $\mathscr{R}_i$ we denote the set of all possible preferences for agent $i$, so that the inclusion $\mathscr{R}_i \subseteq \mathscr{R}_A$ must hold. Consistent with this, $\mathscr{R} = \mathscr{R}_1 \times \ldots \times \mathscr{R}_n$ is then the set of all possible preference profiles. A generic element of $\mathscr{R}_i$ is denoted by $R_i$, a generic element of $\cup \mathscr{R}_i$ is denoted by $R$ and a generic element of $\mathscr{R}$ is denoted by $\boldsymbol{R}$. In a familiar manner, $P$ denotes the strict preference relation corresponding $R$. A *social choice correspondence* (SCC) $f$ is a mapping

$$f : \mathscr{R} \to 2^A,$$

that gives the set of socially optimal alternatives for a given preference profile.

[3] All preferences are complete and transitive.

We shall say that social alternative $a \in A$ is *redundant*, if it does not belong to the range of $f$. This can be expressed as $a \in A \setminus f(A)$. By saying that the planner can *freely use redundant alternatives*, we mean that she can include redundant alternatives in $A$ that enter at any desired position for all the different preferences in the sets $\mathscr{R}_i,\ i = 1, \ldots, n$. If we denote by $d$ some redundant alternative added, then we can uniquely define this planners actions with an indicator function

$$\pi_d : \cup \mathscr{R}_i \rightarrow \ \{0, \ldots, \mid A \mid\},$$

that gives the position where the redundant state $d$ enters in a given preference ordering $R$.[4] We still need to somehow denote the preference we end up after the planners actions, so let $v(\pi_d, R)$ be the preference relation that $R$ turns into after the application of $\pi_d$. It will also be helpfull to have some way to denote the position of an alternative $a \in A$ in a given preference ordering $R$. Let's denote this by $R(a)$.

We shall need one crucial assumption, that is, the adding of redundant alternatives can not affect social choice. Let's define

$$\mathscr{Q}_i = \{v(\pi_d, R) \mid R \in \mathscr{R}_i\}$$

and $\mathscr{Q} = \mathscr{Q}_1 \times \ldots \times \mathscr{Q}_n$. The SCC $\tilde{f}$ in the new set of preference profiles $\mathscr{Q}$ is then defined by the condition

$$\tilde{f}(v(\pi_d, R_1), \ldots, v(\pi_d, R_n)) = f(R_1, \ldots, R_n) \ \ \forall\ \boldsymbol{Q} \in \mathscr{Q}. \tag{1}$$

This can be interpreted to mean that the set $A$ contains all relevant social alternatives when a certain goal represented by $f$ is tried to achieve. No

[4]Since preferences are complete and transitive, we can order the alternatives $A$ in a row according to a decending preference. Indifferent elements can be at any order in the row, but we have to keep track on this (the order can not change once it is fixed). $\pi(R, d) = k$ then means that $d$ is added between $k$th and $(k+1)$th element in this row. Cases $k = 0$ and $k = \mid A \mid$ are interpreted in an obvious way.

more information than the preference orderings on $A$ is needed. We shall be calling $\tilde{f}$ a SCC that is *essentially* $f$ and denote this by $f \sim f$. If $f \sim \tilde{f}$ and $\tilde{f} \sim \hat{f}$, then we say that $f$ is also essentially $\hat{f}$. This makes the relation $\sim$ transitive and allows us to speak about situations in which more than one redundant alternative is added.

Two properties of SCC's are going to be used extensively. Let $L_i(a, \boldsymbol{R})$ denote the lower contour set of agent $i$ at $a \in A$ when the preference profile is $\boldsymbol{R}$ *i.e.*

$$L_i(a, \boldsymbol{R}) = \{b \in A \mid a\boldsymbol{R}_i b\}.$$

A SCC $f$ is called *monotonic*, if for all $\boldsymbol{R}$,$\boldsymbol{R'}$$\in \mathscr{R}$ and $a \in f(\boldsymbol{R})$ the following condition holds

$$L_i(a, \boldsymbol{R}) \subseteq L_i(a, \boldsymbol{R'}) \;\; \forall\, i \in N \quad \Rightarrow \quad a \in f(\boldsymbol{R'}). \tag{2}$$

In words this can be expressed as follows: if alternative $a$ is socially optimal when the preference profile is $\boldsymbol{R}$, then it must also be socially optimal under $\boldsymbol{R'}$ assuming it has not dropped in any agents preference. A SCC $f$ has the property of *no veto power*, if for all alternatives $a \in A$ and all preference profiles $\boldsymbol{R} \in \mathscr{R}$ the following condition hold

$$\#\{i \mid L_i(a, \boldsymbol{R}) = A\} = n - 1 \quad \Rightarrow \quad a \in f(\boldsymbol{R}). \tag{3}$$

In words this can be expressed as follows: if alternative $a$ is top ranked for all agent except one, then it must be socially optimal *i.e* no agent can have veto power.

A tuple $G = (S, g)$, where $S = S_1 \times \ldots \times S_n$ is the set of strategy profiles and $g : S \to A$ is an outcome function giving a social alternative as a function of the strategy profile, is called a *game form* or a *mechanism.* Here $S_i$ is the strategy set of agent $i$. Then, given a strategy profile $\boldsymbol{R}$, the game form $G$ defines a game $\Gamma = \langle G, \boldsymbol{R}\rangle$ in a normal form. A Nash-equilibrium of the game $\Gamma$ is a strategy profile $s \in S$ such that $g(s)\boldsymbol{R}_i g(s_i^*, s_{-i})$ for all $s_i^* \in S_i$

and all $i \in N$. The set of all Nash-equilibria of the game $\Gamma$ is denoted by $NE(\Gamma,\boldsymbol{R})$, to emphasize the dependence on the preference profile $\boldsymbol{R}$. We say that the game form $G$ *Nash-implements* a SCC $f$ if

$$g(NE\,(\Gamma,\boldsymbol{R})) = f(\boldsymbol{R}) \quad \forall\, \boldsymbol{R} \in \mathscr{R}.$$

This means that for every preference profile $\boldsymbol{R} \in \mathscr{R}$, exactly the socially optimal alternatives can be attained as a Nash-equilibrium of the game form $G$. If there exists some game form $G$ that Nash-implements $f$, or some essentially the same correspondence $\tilde{f}$, then we say that $f$ is *Nash-implementable.*

## 3 Examples of Two Agent Case

In the two examples below we shall use the fact, proved in Maskin (1999), that every Nash-implementable choice correspondence must be monotonic. This does not depend on the number of agents involved, even though most of the results in Nash-implementation are different in two agent case. Since this is very central for the rest of the paper, we shall present it here. Let's assume that $g : S_1 \times \ldots \times S_n \to A$ is a game form that implements a choice correspondence $f : \mathscr{R} \to 2^A$ in Nash-equilibrium. Furthermore, let's assume $\boldsymbol{R}$,$\boldsymbol{R'} \in \mathscr{R}$, $a \in f(\boldsymbol{R})$ and that

$$\forall\, i \in N,\; \forall\, b \in A: \quad a\boldsymbol{R}_i b \;\;\Rightarrow\;\; a\boldsymbol{R'}_i b.^5 \tag{4}$$

Because $a \in f(\boldsymbol{R})$ and $f$ is Nash-implementable, there must exist a Nash-equilibrium $s$ such that $g(s) = a$ and $g(s_i, s_{-i})\boldsymbol{R}_i g(s'_i, s_{-i})$ for all $i \in N$ and for all $s'_i \in S_i$. Now if $s$ is not a Nash-equilibrium under $\boldsymbol{R'}$, then there must exist agent $j$ and a strategy $s^*_j \in S_j$ such that $g(s^*_j, s_{-j})\boldsymbol{P'}_j g(s) = a$. But then condition (4) would imply $g(s^*_j, s_{-j})\boldsymbol{P}_j g(s)$, a contradiction with the

[5] This condition is equivalent to (1), which is just expressed using lower contour sets.

fact that $s$ is a Nash-equilibrium under $\boldsymbol{R}$. This means that $s$ must be Nash-equilibrium also under $\boldsymbol{R'}$, which by the definition of Nash-implementability gives us finally $a = g(s) \in f(\boldsymbol{R'})$. This line of argument verifies that all Nash-implementable choice correspondences are also monotonic.

**EXAMPLE 1.** Let's assume we have two agents $i = 1, 2$ and four social alternatives $A = \{a, b, c, d\}$. The preference domain is $\mathscr{R} = \mathscr{R}_1 \times \mathscr{R}_2$, where $\mathscr{R}_1 = \{R_1\}$ and $\mathscr{R}_2 = \{R_2, R_2'\}$. These (linear) preferences are given in table 1 below. Best alternative is on top and the worst on bottom.

| $R_1$ | $R_2$ | $R_2'$ |
|---|---|---|
| $c$ | $b$ | $c$ |
| $a$ | $c$ | $b$ |
| $b$ | $a$ | $a$ |
| $d$ | $d$ | $d$ |

**Table 1**

In this case the preference domain consists of only two profiles and the choice function $f : \mathscr{R} \to A$, defined by $f(R_1, R_2) = a$ and $f(R_1, R_2') = c$, is not Nash-implementable. This follows from the fact that monotonicity would require $f(R_1, R_2') = a$.

Now let's assume that the planner can freely use redundant alternatives. Then she can introduce an alternative $\Delta$, which is presented in table 2, to be used in constructing a game form. Now with the added redundant alternative $\Delta$ the choice function $\tilde{f}$ defined in (1), that is essentially $f$, becomes implementable. This can be verified by noting that the game form in figure 1 implements $\tilde{f}$.

| | $s_2$ | $s_2'$ |
|---|---|---|
| $s_1$ | $a$ | $\Delta$ |
| $s_1'$ | $b$ | $c$ |

**Figure 1**: A game form that implements $\tilde{f}$.

| $v(\pi_\Delta, R_1)$ | $v(\pi_\Delta, R_2)$ | $v(\pi_\Delta, R_2')$ |
|---|---|---|
| $c$ | $b$ | $c$ |
| $\Delta$ | $c$ | $b$ |
| $a$ | $a$ | $\Delta$ |
| $b$ | $\Delta$ | $a$ |
| $d$ | $d$ | $d$ |

**Table 2**

■

In most real world situations, it is quite easy to find social alternatives that enter in different positions of preference orderings of two different agents. What seems to be intuitively more challenging, is to find alternatives that enter in different positions of two preference orderings of the same agent. The case in example 1 is quite abstract, so we must ask whether it has any practical relevance. Next example presents a simple barter economy that can give rise to an exactly analogical resource allocation problem.

**EXAMPLE 2.** Let's assume we have a barter economy with two goods $x$ and $y$, $x_j$ and $y_j$ being the amounts consumed by agent $j = 1, 2$. The planner knows that agent 1 has Leontief preferences *i.e.* $u_1(x_1, y_1) = \min\{x_1, y_1\}$.

Agent 2, on the other hand, has preferences that are either Leontief $u_2'$ or linear $u_2$, but the planner does not know which one. This situation is presented as an Edgeworth box in figure 2, which also contain five possible allocations $a$, $b$, $c$, $d$ and $\Delta$. If we choose $u_1 = R_1$, $u_2 = R_2$ and $u_2' = R_2'$, then these allocations are exactly in the order presented in table 2. In effect, the abstract case in example 1 is not only a theoretical artefact. In a sense the planner can now use the redundant allocation $\Delta$ as an artificial entry in the game form that implements the socially optimal allocation given by $f$.

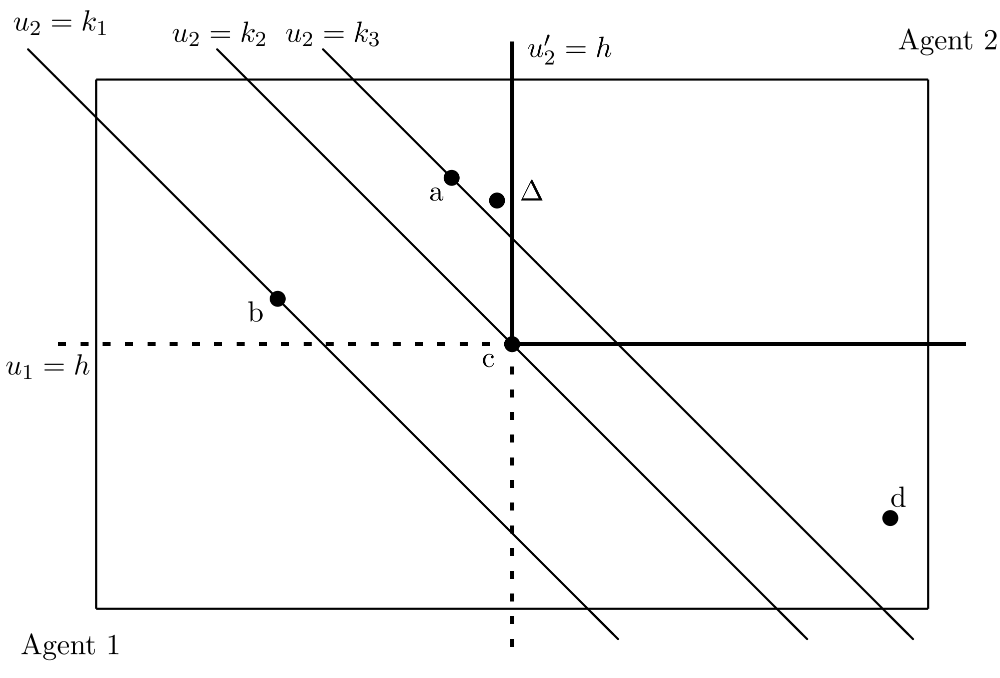


**Figure 2**: An Edgeworth box.

Based on these two examples, it is not very surprising that the *Walrasian -correspondence* and the *Lindahl -correspondence* are Nash-implementable in many settings.[6] The fact is that in resource allocation problems the planner has a lot of possible redundant alternatives at her disposal. Still, these

[6]See Corchón (1996) and references given there.

examples are quite trivial and the problem is bound to get more complicated when the number of possible preference profiles increase. In the next chaper we shall prove that the situation studied in the examples above can be expected to have guite a large generality.

# 4 The Main Result

In the next theorem we shall prove that in practice every SCC $f$ is Nash-implementable, if there are at least three agents.[7] This result is different from that in Abreu and Sen (1991) on virtual implementation, since we claim that in practice every SCC $f$ is Nash-implementable exactly, not only approximately. Furthermore, the result of Abreu and Sen (1991) assumes a metric structure on $A$, which really narrows down the set of applications.

We shall confine ourselves to the more that two agent case, because this allows a simple proof. We belive that the result holds also in the two agent case, which is important, since most contracts are made between two parties.[8]

The proof is based on the result by Maskin (1999), which states that monotonicity (condition 2) and no veto power (condition 3) are sufficient properties for a SCC $f : \mathscr{R} \rightarrow 2^A$ to be Nash-implementable.[9] We shall show that by introducing redundant alternatives in a certain way these properties can be guaranteed to hold. To simplify the presentation, we shall need few notational conventions. If we have two preference profiles $\boldsymbol{R}$, $\boldsymbol{R'} \in \mathscr{R}$ and

[7]What we mean by the phrase "*in practice*" can be fully explained only after the proof. See the remarks below the theorem.

[8]This would mean that every imaginable contract is Nash-enforceable. One shoud notice, that sometimes a contract between two agents can be seen as a three agent case by assuming court as a third party.

[9]Necessary and sufficient conditions are given in Moore and Repullo (1990).

an alternative $a \in A$ such that

$$L_i(a, \boldsymbol{R}) \subseteq L_i(a, \boldsymbol{R'}) \;\; \forall\, i \in N \;\text{ and }\; a \notin f(\boldsymbol{R'}),$$

then we say that the triplet ($\boldsymbol{R}$,$\boldsymbol{R'}$, a) is *an instance of monotonicity failure.* The set that contains all instances of monotonicity failure for $f$ is denoted by MF($f$). In a similar manner, if

$$\#\{i \;\mid\; L_i(a, \boldsymbol{R}) = A\} = n - 1 \;\text{ and }\; a \notin f(\boldsymbol{R}),$$

then we say that the tuple ($R$, a) is *an instance of no veto power failure.* The set that contains all failures of no veto power for $f$ is denoted by NVPF($f$).

**THEOREM.** If $\mid N \mid \geq 3$ and the planner can freely use redundant alternatives, then every social choice correspondence $f : \mathscr{R} \rightarrow 2^A$ is Nash-implementable.

**PROOF.** Let $f : \mathscr{R} \rightarrow 2^A$ be any SCC. Pick some ($\boldsymbol{R}$, a) $\in$ NVPF($f$) and introduce a new redundant alternative $d$ in such way that for some $j \in \{i \;\mid\; L_i(a, \boldsymbol{R}) = A\}$

$$\pi_d(R_i) = \begin{cases} 0, & \text{if } \;\; i = j, \\ \mid A \mid, & \text{if } \;\; i \in N \setminus \{j\}. \end{cases}$$

(the redundant alternative $d$ is now best alternative in every preference of agent $j$ and the worst alternative in every preference of every other agent) This procedure creates a new SCC $\tilde{f}$, that is essentially $f$, but has stricly less cases of no veto power failure. Continuing in this manner we shall finally have a SCC $f'$, that is essentially $f$ and satisfies NVPF($f'$)=$\emptyset$. After this is done, we can continue as follows. Let's denote by $\mathscr{Q}$ the domain of $f'$. This is the set that $\mathscr{R}$ turns into when we apply the aforementioned procedure. Let's also assume, that $D$ is the set of all redundant alternatives added so far. Pick some ($\boldsymbol{Q}$, $\boldsymbol{Q'}$, a) $\in$ MF($f'$) and introduce a new redundant alternative $c$ in such way that for some $j \in N$

$$\pi_c(Q_i) = \begin{cases} Q_j(a), & \text{if} \quad Q_i = \boldsymbol{Q}_j, \\ Q'_j(a-1), & \text{if} \quad Q_i = \boldsymbol{Q'}_j, \\ \text{any possible } k, & \text{if} \quad i = j \text{ and } Q_i \notin \{\boldsymbol{Q}_j, \boldsymbol{Q'}_j\}, \\ \mid A \mid + \mid D \mid, & \text{otherwise.} \end{cases}$$

This procedure creates a new SCC $\tilde{f}'$ that has stricly less monotonicity failures than $f'$ and does not have any no veto power failures. It is clear that this procedure does not create new no veto power failures and NVPF($f'$)=$\emptyset$, so it definitely must be that NVPF($\tilde{f}'$)=$\emptyset$. The reduction in the number of monotonicity failures follows from the fact that for all $\boldsymbol{Q}^*$, $\boldsymbol{Q}^{**} \in \mathscr{Q}$ and all $i \in N$

$$L_i(a, v(\pi_c, \boldsymbol{Q}_i^*)) \subseteq L_i(a, v(\pi_c, \boldsymbol{Q}_i^{**}) \quad \Rightarrow \quad L_i(a, \boldsymbol{Q}_i^*) \subseteq L_i(a, \boldsymbol{Q}_i^{**}),$$

but $L_j(a, \boldsymbol{Q}_j) \subseteq L_j(a, \boldsymbol{Q'}_j)$ and $L_j(a, v(\pi_c, \boldsymbol{Q}_j)) \nsubseteq L_j(a, v(\pi_c, \boldsymbol{Q'}_j))$ by the definitions. Continuing in the same manner we shall end up with a SCC $\hat{f}$, that is essentially $f$ and satisfies NVPF($\hat{f}$)=$\emptyset$ and MF($\hat{f}$)=$\emptyset$. This SCC is then Nash-implementable, since it trivially satisfies monotonicity and no veto power. ***Q.E.D.***

REMARK I. In the proof above, redundant alternatives are used in a very restricted way. In the no veto power case only redundant alternatives that are worst or best for every agent are used. These kind of new alternatives are easy to come up with, since they can be made contingent upon identity. This can be easily achieved by announcing that agent $j$ has to pay (or gain) a certain amount of money in the event this new alternative would be chosen. The case of trivializing monotonicity is a bit more demanding in practice. The only thing that is required though, is that the planner can use some new alternative which is below $a$ in the preference $\boldsymbol{Q}_j$ and above it in the preference $\boldsymbol{Q'}_j$. How this alternative enter in the other preferences of $j$ does not matter. Furthermore, the planner can again make this same alternative contingent on identity, so that the idea mentioned previously apply.

REMARK II. The crucial thing to notice is that redundant alternative can be anything imaginable. This follows from the definition of Nash-implementation, which guarantee that redundant alternatives never actually become chosen. This is because thay are not in the range of the social choice correspondence. Redundant alternatives are only used to construct a game form in which they are never equilibria. In an economic environment this would mean that there is no danger of budget imbalace, no matter what kind of redundant allocations are used.

REMARK III. The procedure used in the theorem does not work, if NVPF($f$)=$\infty$ or MF($f$)=$\infty$ or both. These conditions are always met if $\mid A \mid < \infty$, but not necessarily if $\mid A \mid = \infty$.

## 5 Conclusion

For a long time it has been a kind of unwritten conception, that Nash-implementation is too permissive concept. No one has tried to give any deeper explanation on why this is so, which has been the main purpose of this paper. We have argued that by giving the planner a very restricted capabilities for social engineering, all choice correspondences become Nash-implementable. Furthermore, this result is not only an approximative one, as in the literature on virtual implementation, but exact. The basic driving force behind it is the fact, that in practice new redundant alternatives showing up in the game form but not in the range of the choice correpondence can usually be invented. This leaves one fundamental question open, which can not be aswered purely on thoretical grounds and has not been adressed here. Is Nash-equilibrium still a good prediction after new redundant alternatives are added? This will pose some further restricions on the redundant alternatives that can be used, but ultimately it is again just a question of social engineering.

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