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Congestion, Coordination and Matching

Aboa Centre for Economics

Discussion Paper No. 28
Turku 2008

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ISSN 1796-3133

Turun kauppakorkeakoulun monistamo
Turku 2008

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January 2008

ABSTRACT

We study the existence of pure strategy Nash equilibria in finite congestion and coordination games. Player set is divided into two disjoint groups, called men and women. A man choosing an action a is better off if the number of other men choosing a decreases, or if the number of women choosing a increases. Analogously, a woman becomes better off if more men or fewer women choose the same action as she does. Existence proofs are constructive: we build simple ”best reply” algorithms that converge to an equilibrium.

JEL Classification: C70, C72, C78, D83

Keywords: congestion, coordination, matching

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Acknowledgements

We thank Klaus Kultti and Hannu Vartiainen for useful comments and discussions.

1 Introduction

We study the existence of pure strategy Nash equilibria in finite congestion and coordination games. Player set is divided into two disjoint groups, called men and women. A man choosing an action a becomes better off if the number of other men choosing a decreases, or if the number of women choosing a increases. Analogously, a woman becomes better off if more men or fewer women choose the same action as she does. In addition, there could exist *pairwise congestion* or *pairwise coordination* effects depending on whether a player becomes worse or better off when more *couples* (consisting of a man and a woman) choose the same action as (s)he does. Existence proofs are constructive: we build simple "best reply" -algorithms that converge to an equilibrium.

As an example, we can think of a group of women and men choosing among the several bars or restaurants in a city. The decisions of the men and women do not solely depend on the characteristics of bars (wine lists, type of music played, *etc.*) but also on the number of other men and women coming to the same bar. Women are the better off the more men choose the same bar and the worse off the more other women choose the same bar, for men *vice versa*. Or, the players could be males and females of some other species searching for feeding and breeding grounds (see Milinsky (1988)).

For another example, think about actions being holiday resorts and players being consumers and firms. Firms decide where to build a hotel (or other facilities) and customers choose where to spend their holiday. Competition among firms in the same location is good for the customers, but they don't like the area being too crowded. Or firms could be deciding in which tv -channel or newspaper to advertize and customers could be people deciding which channel to watch or which newspaper to buy.

Technically our model is closest to Milchtaich's (1996) and Quint and Shubik's (1994) model. In their model, there are only congestion effects, or in our terminology, all players are men. It is shown in both of these papers that there is a very simple algorithm converging to equilibrium. The players are introduced in the game one by one. Each new player chooses an action to maximize his utility given the actions chosen by the players already in the game. When a new player has made his move, old players may revise their actions. The revision phase continues as long as there are any players who want to deviate. If all players are satisfied in their current actions, a new player enters and chooses an action, and so on, until an equilibrium is found.

We use the following algorithm for games with *pairwise congestion* property: First we form as many man - woman pairs as possible, and leave the remaining players (men) single. Single players choose first following the Milchtaich or Quint-Shubik -algorithm. After that the couples enter the game one by one. The woman makes the first choice for the couple. In the revision phase, men and women may change their actions individually (so the originally "married men" need no longer follow the orders of their "wives"), but in the opposite order they entered the game. This algorithm is modified slightly for games with *pairwise coordination* property.

Rosenthal (1973) was the first to define the class of congestion games and to prove the existence of Nash equilibrium (his definition is slightly different than the one adopted by Milchtaich (1996) and Quint and Shubik (1994)). Monderer and Shapley (1996) introduced the class of potential games, which includes Rosenthal's congestion games, and proved that these games admit a pure strategy equilibrium. Konishi, Le Breton and Weber (1997a) proved the existence of a strong pure strategy equilibrium in the class considered by Milchtaich (1996) and Quint and Shubik (1994). Konishi *et.al.* (1997b) defined a class of games with *positive externalities* and proved the existence of pure strategy equilibrium in this class by constructing a potential function.

In our model men get positive externalities from women and *vice versa*, but there are negative externalities (congestion effects) within groups. We are not aware of other papers where the existence pure strategy equilibria is proved in models exhibiting both positive and negative externalities.

The paper is organized in the following way. In Section 2 notation is introduced. The main results are presented in Section 3, and Section 4 contains examples.

2 Preliminaries

Let $G = \{N; (S_i)_{i\in N}; (u_i)_{i\in N}\}$ be a finite normal form game. That is, the set N is a finite set of *players*, S_i is a finite set of *strategies* of player i, and $u_i : \prod\limits_i S_i \longrightarrow \mathbb{R}$ is the *utility function* of player i. We assume that $S_i = A$ for all $i \in N$ and denote a strategy profile by s, $s \in A^N$. The game has two types of players, that is $N = M \cup W$, where M and W are nonempty disjoint sets. We will call players in M and W as men and women, respectively.

Let $n_s(a)$ be the number of players who chose an action a in a strategy profile s and let $m_s(a)$ and $w_s(a)$ be the number of players $i \in M$ and $j \in W$, respectively, who chose an action a in a strategy profile s. We assume that there are functions $u : A \times \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}$ and $v : A \times \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}$ such that

$$u_i(s) = u(a \,|\, (m_s(a), w_s(a))), \text{for all } i \in M \text{and } s \text{ such that } s_i = a,$$

$$u_i(s) = v(a \,|\, (m_s(a), w_s(a))), \text{for all } i \in W \text{and } s \text{ such that } s_i = a.$$

To get simpler notation, we will in the sequel denote the utility of a man from the action a by $u(a \,|\, (m, w))$, where m is the number of men and

w is the number of women choosing a. Similarly, the utility of a woman from the action a is denoted by $v(a \,|\, (m, w))$. A game G is now given by $G = \{A; M, u; W, v\}$.

There are several assumptions made when utilities have this kind of neat expression. First, player $i's$ utility from s does not depend on the identity of his opponents j making choices $s_j = b$, but it may depend on whether j is a man or a woman (*anonymity inside groups*, see Konishi *et.al.* (1997a)). Secondly, player's utility from an action a depends only on the number of players from each group choosing this same action (*independence of irrelevant choices*, see Konishi *et.al.* (1997a)). Thirdly, all women have the same utility function and all men have the same utility function (*symmetry inside groups*).

The next assumption is called *population monotonicity (PM)*. We assume that the payoff of any man i (any woman j) decreases when more men (women) choose the same action as i (j) (*negative population monotonicity NPM*) and increases when more women (men) choose the same action as i (j) (*positive population monotonicity PPM*).

Definition 1 *A game G has the PM-property if for all $a \in A$ the conditions NPM and PPM hold:*
(NPM) if $m' > m$ then $u(a \,|\, (m, w)) \geq u(a \,|\, (m', w))$, if $w' > w$ then $v(a \,|\, (m, w)) \geq v(a \,|\, (m, w'))$;
(PPM) if $w' > w$ then $u(a \,|\, (m, w')) \geq u(a \,|\, (m, w))$, if $m' > m$ then $v(a \,|\, (m', w)) \geq v(a \,|\, (m, w))$.

The following condition is called *pairwise congestion (PCG)*. If a player prefers a to b, and one man and one woman join the group who chooses b, then a is still preferred to b.

Definition 2 *A game G has the PCG -property, if $u(a \,|\, (m, w)) \geq u(b \,|\, (m', w'))$*

implies $u(a \,|\, (m,w)) \geq u(b \,|\, (m'+1, w'+1))$*, and if* $v(a \,|\, (m,w)) \geq v(b \,|\, (m', w'))$ *implies* $v(a \,|\, (m,w)) \geq v(b \,|\, (m'+1, w'+1))$*, for all* $a \in A$*.*

The following condition is called *pairwise coordination (PCD)*. If a player prefers a to b, and one man and one woman join the group who chooses a, then a is still preferred to b.

Definition 3 *A game* G *has the PCD -property, if* $u(a \,|\, (m,w)) \geq u(b \,|\, (m', w'))$ *implies* $u(a \,|\, (m+1, w+1)) \geq u(b \,|\, (m', w'))$*, and if* $v(a \,|\, (m,w)) \geq v(b \,|\, (m', w'))$ *implies* $v(a \,|\, (m+1, w+1)) \geq v(b \,|\, (m', w'))$*, for all* $a \in A$*.*

The condition PCG implies that a suboptimal action for a player cannot become an optimal one if a new couple chooses this action. The condition PCD implies that an optimal action stays optimal if a new couple chooses this action. The condition PCG (or PCD) is therefore less demanding than a monotonicity condition that requires that player's utility decreases (increases) if a new couple chooses his/her action.

3 Results

We consider first games with *population monotonicity* and *pairwise congestion* properties. It turns out that in such games there are pure strategy Nash equilibria. We assume w.l.o.g. that there are at least as many men as there are women. Consider the following algorithm for games $G = \{A; M, u; W, v\}$:

THE ALGORITHM. Index the players by natural numbers so that the first $|W|$ odd natural numbers are indices for women and the first $|M|$ positive even natural numbers are indindices for men. So the highest index for a woman is $2|W| - 1$ if there are any women, and the highest index for a man is $n = 2|M|$. We describe the algorithm separately for the cases when there are (A) no women, (B) both men and women and $|M| \geq |W|$.

(A) Choose an action a for man $i = 2|M|$ that maximizes his utility $u(a\,|\,(1,0))$. Suppose each i among the $k \geq 1$ men with highest indices have given some action $a(i)$ that maximizes his utility $u(a(i)\,|\,(m,0))$ when in total m men choose this action. Then give the man $j = 2|M| - 2k$ an action $b(j)$ that maximizes his utility $u(b(j)\,|\,(m,0))$ when in total m men choose this action. Next let the men revise their actions one at a time so that at each stage the man with the lowest index is given the opportunity to change his action first. Only those men for whom there are actions that are strictly better than their current choice are allowed to revise. When looking for better actions, all players think myopically that after his choice there will be no more revisions. If at some point actions for all men $i = 2|M| - 2k, \ldots, 2|M|$ are such that none of them wants to revise any more, then the next man $j = 2|M| - 2k - 2$ is given an action $b(j)$ that maximizes his utility $u(b(j)\,|\,(m,0))$ given that in total m men choose this action. The algorithm converges if and only if all men have assigned an action and none of them wants to change his action any more.

(B) First all the "single" men $2|W|+2, \ldots, 2|M|$ (if $|M| > |W|$) are given an action by using the algorithm (A) (if this algorithm stops), and after that the "couples" $(2|W|-1, 2|W|), \ldots, (1,2)$ are introduced in the game in that order. New couples are brought in the game only if none of the old players wants to change his or her action. When the couple $(i, i+1)$ enters, the woman i chooses the action x that maximizes her utility $v(x\,|\,(m+1, w+1))$, where m and w are number of men and women choosing x before the couple $(1,2)$ entered the game. The revision stage is again such that the player with lowest index may first revise his or her action if he or she strictly gains by doing so. In particular, the man $i+1$ need no longer choose the same action that was originally chosen by his "wife" i. The algorithm converges if and only if all players have assigned an action and none of them wants to change his or her action any more.

It is clear that if the algorithm converges, and the resulting strategy profile is s, then s is a Nash equilibrium. Our main theorem gives sufficient conditions for the convergence for generic games. We call a game $G = \{A; M, u; W, v\}$ *generic*, if $u(a \,|\, (m, w)) \neq u(a' \,|\, (m', w'))$ if $(a, m, w) \neq (a', m', w')$ and similarly for the women's utility function v.

Theorem 1 *If a generic game G has the properties PM and PCG, then the algorithm converges to a Nash equilibrium.*

Proof. Convergence of the case (A) of the algorithm follows from Theorem 2 of Milchtaich (1996) and Theorem 3 of Quint and Shubik (1994). The algorithm actually converges in our case always immediately without any action revisions by any man. We give the proof for completeness.

(A) Let G be a game having properties PM and PCG in which there are M men. If there is only one man, let a be a choice that maximizes the utility $u(x \,|\, (1, 0))$. Because there are no other men, there will be no revisions.

Suppose then that the algorithm converges always when there are at most $k-1 \geq 1$ men, and that the convergence happens always immediately without anybody wanting to revise his action.

If there are k men in the game G, then by the induction assumption the man with index $i = 2$ is brought into the game at some stage. Let a be the action that maximizes the utility $u(x \,|\, (t, 0))$ of the man 2, given the choices of the old players. Then if a player $i > 2$ has chosen an action

$b \neq a$, this action is still utility maximizing by *PCG*. None of the men who chose a would like to change her action, since a maximizes the utility of the man 2 and all men have identical utility functions. Therefore the algorithm converges immediately without any revisions.

(B) Let G be a game having properties *PM* and *PCG* such that $|M| \geq |W| > 0$. We proceed by induction in the number of "couples" $(i, i+1)$, i is a woman and $i+1$ is a man.

Suppose that there are k couples and any finite number of single men. By the case (A) above, the algorithm applied to the single men converges. Assume now that the algorithm converges when there are at most $k-1 \geq 0$ couples $(i, i+1)$.

Denote by $(1, 2)$ the k'th couple that is introduced in the game, when all the other players are choosing equilibrium actions, $k \geq 1$. (The proof is the same whether or not $(1, 2)$ is the *only* couple in the game.) So let a be the action that maximizes $v(x \,|\, (m+1, w+1))$ where m and w are the number of men and women choosing x *before* 1 and 2 enter the game.

In the revision phase, the player with the lowest index who finds a strictly better action may change his or her action first. By *PCG*, none of the players choosing $b \neq a$ wants to change his or her action, and clearly no woman choosing a wants to change her action either. So the only players who possibly could gain by changing action are the men choosing a. The algorithm gives the man 2 the first chance to revise his action, and suppose that he finds action b strictly better than action a and deviates to b.

The man 2 thus has moved from a to b. No other man wants to deviate: the only possibility would be some man wantig to choose a, but then this same man should have chosen b before the couple $(1, 2)$ entered the game. So only some women might want to change action. The woman 1 is asked first if she can strictly increase her utility. Three cases arise: (i) woman 1 does not change her action, (ii) woman 1 moves to b also, and (iii) woman 1 moves from a to $c \neq b$.

(i) Woman 1 does not change her action:

If the woman 1 cannot increase her utility the algorithm has converged. To see this, notice that no man with action a or b can increase his utility by changing action. So if some man wants to revise, it must be some man $i > 2$ who, at the moment, chooses an action $c \neq a, b$. The only action that could be strictly better than c for him is a. But the man 2 just moved from a to b, so b was strictly better than c for man i prior to the introduction of the couple $(1, 2)$ to the game, a contradiction. For women $i \neq 1$, the only possible new action is b. But if there are such women, then woman 1 wants to deviate to b as well. Namely, if this were not true, then the woman $i \neq 1$ should have chosen a just before the couple entered the game rather than her current action. So if woman 1 doesn't want to deviate to *any* action, the algorithm converges and woman 1 chooses a and man 2 chooses b in equilibrium.

(ii) Woman 1 moves to b also:

Suppose then that the woman 1 *can* strictly increase her utility after man 2 has moved from a to b. If woman 1 wants to choose b rather than a then she can make this move before any other women. By PCG and induction assumption no man $i > 2$ choosing $c \neq b$ wants to choose b. Similarly by induction assumption, no man $i > 2$ who chose $c \neq a$ wants to deviate to a. The choice b maximized the utility of man 2 *before* woman 1 moved there, and therefore b by PM his utility only increases when woman 1 moves to b. This holds true for all men choosing b since they have identical utility functions, and therefore the algorithm converges.

(iii) Woman 1 moves from a to $c \neq b$:

In this case, no other woman wants to change her action any more, the reason being the same as in case (i). However, there could be some man wanting to choose c as well. We will show that in this case man 2 also wants to move from b to c. Namely, if this doesn't hold, then by genericity of the game, man 2 would strictly prefer b to c at this moment. Let $k \geq 0$ be the number of men choosing c, $m \geq 1$ the number of men and choosing b, and $h \geq 1$ the number of men choosing x at the moment. Let $w_1, w_2, w_3 \geq 0$ be the number of women choosing c, x, and b at the moment. If a man wants to change from x to c, the following inequality must hold:

$$u(c \,|\, (k+1, w_1)) > u(x \,|\, (h, w_2)) \tag{1}$$

On the other hand, if the man 2 does not want to change from b to c,

$$u(b \,|\, (m, w_3)) > u(c \,|\, (k+1, w_1)) \tag{2}$$

But the two inequalities imply that the man who wanted to change from x to c, had chosen b and not x before the couple $(1, 2)$ entered the game, a contradiction.

To sum up, if there is at least some man who wants to move from x to c after the woman 1 moved from a to c, then also the man 2 wants to move from b to c. Since man 2 is allowed to change his action first, he will in fact move from b to c.

But now the algorithm converges, since all the single man who didn't choose c before the couple $(1, 2)$ entered the game, now find c even worse alternative by *PCG*. Since c is a best choice for 1 and 2, no player choosing c wants to revise his or her action. But in such a case man 2 wants to deviate to c as well, and if man 2 wants to deviate to c, then he makes this move before any other men. The algorithm then converges to equilibrium where the couple $(1, 2)$ chooses c. If man 1 doesn't want to choose c, then the algorithm converges to equilibrium where woman 1 chooses c and man 2 chooses b. This completes the proof. ■

The algorithm doesn't seem to require excessive computational or predictive powers from players: players (or couples) enter the game sequentially, see what choices the other players have made so far, and based on that make their utility maximizing choices. Players do not try to predict the future: they do not try to guess what those players will do that enter the game later, or how the old players will react when new players enter the game. The following result shows that convergence is quite fast despite of players behaving in such a myopic manner.

Proposition 1 *Given a generic game G satisfying properties PM and PCG with $|M|$ men and $|W|$ women, $|M| \geq |W|$, the algorithm converges in at most $3|W| + |M|$ steps.*

Proof. As long as there are only single men in the game, no man ever wants to revise his action he took when he entered the game. So it takes $|M| - |W|$ steps before the first couple is introduced in the game. When the first couple arrives, the longest path before the next couple is brought to the game is the following: the woman 1 chooses a for the couple $(1, 2)$, then the man 2 deviates to $b \neq a$, then the woman 1 deviates to $c \neq a, b$, and finally the man 2 deviates to c (see part (B) in the proof of Theorem 1). So it takes at most $|M| - |W| + 4$ steps before the second couple ienters the game. But it takes at most four steps for every new couple $(i, i + 1)$ before they both are satisfied with their actions and before a new couple enters the game (see part (B) in the proof of Theorem 1). Since there are $|W|$ couples, the algorithm converges in at most $|M| - |W| + 4|W| = 3|W| + |M|$ steps. ■

We will next analyze games in which the *pairwise congestion* (*PCG*) is replaced by *pairwise coordination* (*PCD*). We don't know whether or not our original algorithm converges in these kind of games, but by modifying the algorithm slightly, we can quarantee convergence.

THE MODIFIED ALGORITHM. Index players as in the original algorithm, and assume w.l.o.g. that $|M| \geq |W|$. Index also the actions by the first k natural numbers. The stage (A) is exactly as it was before: the single men choose in the reverse order of their indices, and the revision phase is as before.

If there are some woman, then the stage (B) is divided into two substages B1 and B2.

(B1) Given the choices of single men, check if there is any action a such that if all couples choose this action, then no single woman or man wants to deviate unilaterally. If there are several such actions, choose the one that is best for women. If the set of these optima for women is not singleton, choose the one from this set that is best for men. If this does not resolve ties, then choose the optimum that has the lowest index. If there are no action a such that if all couples choose this action, no single woman or man wants to deviate unilaterally, then move to B2.

(B2) This stage is the same as stage B in the original algorithm.

Again, if the modified algorithm converges, the resulting strategy profile is a Nash equilibrium. The next result says that the modified algorithm converges for generic games satisfying *PM* and *PCD*.

Theorem 2 *If a generic game G has the properties PM and PCD, then the modified algorithm converges to a Nash equilibrium. In this equilibrium either all couples choose the same action, or there is no action chosen both by men and women.*

Proof. The case (A) is the same as in Theorem 1 since there are no woman.

Let G be a generic game having properties *PM* and *PCD* such that $|M| \geq |W| > 0$.

(B1) Suppose there is an action such that when all couples choose this action (plus the single men who chose this action in stage A), then no single man or woman wants to deviate from a. By genericity of the game, there is a unique best such action for women, a. Since no player choosing a wants to change action, the only potential deviators are men choosing some $b \neq a$. The first such deviator is a man with the lowest index i, and his new choice must be a. After that, no player choosing a has any incentives to deviate, so again the only potential deviators are men *not* choosing a. The next deviator could choose either a or the action that the first deviator chose before the couples entered the game.

Continuing the revision stage, we will show that there will never be such an instance that some of the players choosing a would like to change action. Suppose to the contrary. Then there is also the first instance when a player choosing a wants to deviate.

Assume first that this player is a man. Then since the players choosing a have the lowest indices, the man 2 may change his action from a to his new optimum b. But this is possible only if some man has just changed his action from b to some $c \neq a$. But then man 2 had changed his action alredy in the previous stage, a contradiction.

Assume then that the first player wanting to deviate from a is a woman. Again, woman 1 may change her action as soon as she finds a new optimum b. The only reason why b is now better than a for her is that there are now more men choosing b than there were immediately after stage A. Then the men choosing b get a strictly lower utility than the men who chose b

immediately after stage A. But this is a contradiction with PM: there are now at most as many men choosing actions $x \neq a$ as there were immediately after stage A, so there is no reason why more men should be choosing b now than immediately after stage A. But then the algorithm converges, since the game is now essentially a game between single men only. Clearly all couples choose the same action.

Suppose therefore that there is no action such that when all couples choose this action (plus the single men who chose this action in stage A), then no single man or woman wants to deviate from a, and move to the next stage.

(B2). The stage A has converged, the stage B1 has failed, and therefore the couples enter the game one by one just like in stage B of the original algorithm. When the first couple enter, it chooses the action a that maximizes the utility of woman 1. Then man 2 must find a strictly better action b, since otherwise a would by PCD be an action that guarantees the convergence already in stage B1. After man 2 has chosen b, woman 1 may find another best action c, but again $c = b$ is not possible. For the same reason, the best action b or c for woman 1 cannot be chosen by any other man $i > 2$ either. Therefore, all players must be satisfied with their actions as soon as woman 1 and man 2 have found their best actions.

Excatly the same argument applies at each stage when a new couple enters the game, so the algorithm converges, and there is no action chosen both by men and women. ■

Theorems 1 and 2 establish the convergence of the algorithm and hence the existence of a pure strategy Nash equilibrium in *generic* games. In non-generic games where there may be multiple best replies to a given strategy profile, the algorithm should be redefined so that the action taken at each step is uniquely defined. But it is not clear how to redefine the algorithm in such a way that it converges. In fact it is not clear that any such redefined algorithm converges. The following result establishes that in all games satisfying either *PM* and *PCG*, or *PM* and *PCD*, the algorithm can be redefined in such a way that it converges to an equilibrium.

Proposition 2 *Given a non-generic game G satisfying either PM and PCG, or PM and PCD, the algorithm can be redefined in such a way that at each step of the algorithm, the action chosen is uniquely defined, and the algorithm converges to a pure strategy Nash equilibrium.*

Proof. Suppose that G satisfying *PM* and *PCG* is generic otherwise except that are two profiles (a, m, w) and (a', m', w') such that $u(a \mid (m, w)) = u(a' \mid (m', w'))$. Construct a new game G^1 by breaking this indifference by adding (or subtracting) a sufficiently small amount ϵ of utility to $u(a \mid (m, w))$ (or from $u(a \mid (m, w))$) in such a way that G^1 satisfies *PM* and *PCG*, and that the other strict preferences in games G and G^1 are the same. Clearly this can be done and G^1 is generic.

Define G^n like the game G^1 except that ϵ is replaced by ϵ/n, for $n = 1, 2 \ldots$. Then each G^n is generic, and there is a Nash equilibrium $s(n)$ selected by the algorithm for each n. Since there are only finitely many actions, there is naturall number K such that (i) $s(n) = s(K)$ for all $n \geq K$ and (ii) the algorithm produces an identical path to the equilibrium for all $n \geq K$.

Then $s^* = s(K)$ is a pure strategy equilibrium of the game G. Consider all the ways one can define different version of the algorithm by defining the choice in a different manner when there are indifferencies. There are only finitely many such versions since there are only finitely many actions and players. At least one of them produces for game G the same path to the equilibrium $s^* = s(K)$ as the original algorithm produces for the game G^K.

One can show by induction that if there are in G many instances where profiles (a, m, w) and (a', m', w') are indifferent, then again one can construct in the similar manner a sequence of generic games converging to G. The claim holds of course also if G and the generic games satisfy PM and PCD. ■

4 Examples

We assumed that inside groups all players are similar, that is, all women have the same utility function and all men have the same utility function. Next example shows that if we drop this assumption, the game may not have a Nash equilibrium.

Example 1 Let G be a three-person game with action set $A = \{a, b, c\}$, and $W = \{1\}$ and $M = \{2, 4\}$. Player 1 is a woman, and players 2 and 4 are men. Action c is strictly dominated for players 1 and 2, and action b is strictly dominated for player 4. The utility function for player 1 , the woman, satisfies:

$$v(b \,|\, (1,1)) > v(a \,|\, (1,1)) > v(b \,|\, (0,1)).$$

The utility function for player 2, one of the two men, satisfies:

$$u_2(a \,|\, (1,1)) > u_2(a \,|\, (1,0)) > u_2(b \,|\, (1,1)) > u_2(b \,|\, (1,0)) > u_2(a \,|\, (2,1)).$$

The utility function for player 4, the other of the two men, satisfies:

$$u_4(a \,|\, (1,1)) > u_4(a \,|\, (2,1)) > u_4(c \,|\, (1,0)) > u_4(a \,|\, (1,0)).$$

These functions can easily be extended to all action-(m, w) -pairs in such a way that the condition PM is satisfied. Note that player 1 must choose either a or b in every Nash equilibrium. If she chooses a, then 2 must choose b, since 4 will definitely choose a. But then 1 would like to deviate to b, so there is no Nash equilibrium where 1 chooses a. If 1 chooses b, then 4 will never choose a, and therefore 2 chooses a and 4 chooses c. But then 1 would like to deviate to a, so there is no Nash equilibrium where 1 chooses b. Therefore there exists no Nash equilibria in the game G.

Next we study the efficiency of the equilibrium produced by the Algorithm. The next example shows that this equilibrium need not be efficient, in fact, there may be another equilibrium in the game that Pareto dominates the one selected by the Algorithm.

Example 2 Let G be a two person game with $A = \{a, b, c, d\}$, $W = \{1\}$ and $M = \{2\}$. The functions are as follows:

$$\begin{aligned}
v(a \,|\, (0,1)) &= 1 & u(a \,|\, (1,0)) = 0 \\
v(a \,|\, (1,1)) &= 7 & u(a \,|\, (1,1)) = 1 \\
v(b \,|\, (0,1)) &= 0 & u(b \,|\, (1,0)) = 3 \\
v(b \,|\, (1,1)) &= 3 & u(b \,|\, (1,1)) = 4 \\
v(c \,|\, (0,1)) &= 4\frac{1}{2} & u(c \,|\, (1,0)) = 2 \\
v(c \,|\, (1,1)) &= 5 & u(c \,|\, (1,1)) = 2\frac{1}{2} \\
v(d \,|\, (0,1)) &= 2 & u(d \,|\, (1,0)) = -1 \\
v(d \,|\, (1,1)) &= 6 & u(d \,|\, (1,1)) = 5
\end{aligned}$$

These functions have the properties PM and PCG. Applying the Algorithm, first player 1 chooses for the couple $\{1, 2\}$ action a. Next player 2 gets chance to revise.his action. He can increase his utility, so he deviates and chooses b. Player 1 now gets 1 by choosing a so she also wants do revise her action. She chooses c. Player 2 no longer can increase his utility and the Algorithm thus stops. In this equilibrium, player 1 gets utility $4\frac{1}{2}$ and player 2 gets utility 3. However, if both players choose d, then this is also an equilibrium, and player 1 gets utility 6 and player 2 gets utility 5.

References

Konishi H., Le Breton M., and Weber S. (1997a). Equilibria in a Model with Partial Rivalry. *Journal of Economic Theory* **72**, 225-237.

Konishi H., Le Breton M., and Weber S. (1997b). Pure Strategy Nash Equilibria in a Group Formation Game with Positive Externalities. *Games and Economic Behavior* **21**, 161-182

Milchtaich I. (1996). Congestion Games with Player-Specific Payoff Functions. *Games and Economic Behavior* **13**, 111-124.

Milchtaich I. (1998). Crowding Games are Sequantially Solvable. *International Journal of Game Theory* **27**, 501-509.

Milinsky, M. (1988). Games Fish Play: Making Decisions as a Social Forager. *TREE* **3**, 12: 325-330.

Monderer, D., and Shapley, L. (1996). Potential Games. *Games and Economic Behavior* **13**, 124-143.

Quint T., and Shubik, M. (1994) A Model of Migration. Working Paper, Cowless Foundation, Yale University.

Rosenthal, R.W. (1973) A Class of Games Possessing Pure-Strategy Nash Equilibria. *International Journal of Game Theory* **2**, 65-67.

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ISSN 1796-3133