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ABSTRACT

We investigate within a continuous time setting how Knightian uncertainty characterized by κ-ignorance affects the optimal timing policies of a risk-neutral and uncertainty averse investor in the case where the exercise payoff is monotonic. We prove that increased Knightian uncertainty unambiguously decreases the value of the optimal timing policy of an uncertainty averse investor. We also show that higher Knightian uncertainty accelerates timing by shrinking the continuation region whenever the termination payoff is independent of Knightian uncertainty. If this independence condition is not fulfilled, then our results indicate that higher Knightian uncertainty may decelerate optimal timing.

JEL Classification: C61, D81, D92

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1 Introduction

One of the basic qualitative conclusions of the real options literature on the optimal exercise of deferrable investment opportunities is that under risk neutrality increased uncertainty should decelerate the optimal timing of irreversible decisions by increasing the required rate of return and expanding the continuation set where exercising the opportunity is suboptimal. The reason for this observation is that even though increased uncertainty may have a positive impact on the expected present value of the payoff received at exercise it simultaneously increases the value accrued by postponing the decision further into the future (cf. Dixit and Pindyck (1994)). Since the latter of these two counteracting effects dominates the former, increased uncertainty tends to decelerate the optimal timing of irreversible decisions. This intuitively appealing result is, however, based on the assumption that from the point of view of the investor all the relevant probabilistic information affecting the irreversible decision can be summarized into a single probability measure characterizing completely the intertemporal stochastic behavior of the underlying state dynamics.

The aforementioned approach to irreversible decision making has been recently challenged on the basis of the original work by Knight (1921) and the subsequent work by Ellsberg (1961). According to Knight (1921) economic decision makers face two type of uncertainties; namely, measurable and unmeasurable uncertainty. Measurable uncertainty is typically labeled as risk and it can be analyzed on the basis of a single probability measure characterizing the stochastic behavior of the underlying factors affecting economic decisions. The second type of these uncertainties, which is typically called either Knightian uncertainty or ambiguity, cannot be measured and it can be interpreted as the degree of uncertainty on the plausibility or credibility of a particular probability measure characterizing the probabilistic structure of the alternative states of the world. Since the latter of this types of uncertainties is more prevalent in economic decision making, studies focusing solely on risk overlook a key factor affecting real economic decision making problems. In light of the Ellsberg Paradox, the distinction of these two types of uncertainties is important as decision makers prefer to act on the basis of known probabilities rather than on the basis of ambiguous or unsure probabilities (Ellsberg (1961) and Epstein and Wang (1994)). Based on the pioneering work by Knight (1921) and the subsequent research by Ellsberg (1961) ambiguity was first rigorously axiomatized in a atemporal multiple priors setting by Gilboa and Schmeidler

(1989) (for further extensions and refinements, see also Epstein (1999), Ghirardato and Marinacci (2002), and Maccheroni et al. (2006)). Their axiomatization was subsequently extended into an intertemporal recursive multiple priors setting in Chen and Epstein (2002) and Epstein and Schneider (2003) (see also Epstein and Wang (1994)).

The impact of Knightian uncertainty on optimal timing decisions was first investigated by Nishimura and Ozaki (2004) in a job search model. The impact of Knightian uncertainty on the optimal investment timing decisions were subsequently investigated by Nishimura and Ozaki (2007) in a continuous time model based on geometric Brownian motion and in Miao and Wang (2007) in a model based on a general discrete time Feller-continuous Markov process. An important conclusions of these studies is that Knightian uncertainty affects irreversible decisions in a way which radically differs from the impact of risk. Even though increased risk tends to decelerate optimal timing, increased ambiguity may lead to a completely opposite conclusion. The reason for this observation is that higher Knightian uncertainty decreases the confidence of the decision maker on the credibility of the probability distribution describing the stochastic behavior of the underlying state variable. Consequently, a rational decision maker becomes more reluctant to postpone the timing of the decision further into the future on the basis of this potentially more biased probability distribution (Nishimura and Ozaki (2007)). Miao and Wang (2007) also showed that the impact of higher ambiguity on the optimal timing policy depends heavily on the intertemporal specification of the exercise payoff. More precisely, they proved that the impact of Knightian uncertainty on the optimal timing decision depends on the relative degrees of ambiguity about continuation and termination payoffs (Miao and Wang (2007)). An important implication of their findings on the optimal timing of exit is that higher Knightian uncertainty may lead to the optimality of the standard myopic NPV-rule for extremely high degrees of ambiguity aversion.

The objective of this study is to extend part of the analysis of Miao and Wang (2007) to the continuous time setting and investigate how Knightian uncertainty affects the optimal timing policies of an uncertainty averse and risk neutral investor in the case where the underlying state variable is modeled as a one dimensional diffusion and both the continuation and the termination payoffs are monotone. We assume that Knightian uncertainty is characterized by κ-ignorance and, therefore, ambiguity is introduced by assuming that instead of facing a single probability measure the investor faces a continuum of equivalent measures defined by a parameterized family of likelihood ratios (cf. Chen and Epstein (2002) and Nishimura and Ozaki (2007)). Along the lines of the findings by Miao and Wang (2007) and Nishimura and Ozaki (2007), we find that the sign of the impact of increased Knightian uncertainty on the value of the optimal timing policy of an uncertainty averse investor is unambiguously negative. The same conclusion is shown to be valid for the value of arbitrary single threshold policies. However, as indicated by the results of Miao and Wang (2007), the impact of increased Knightian uncertainty on the optimal timing policy is proven to depend on the precise inter-temporal specification of the exercise payoff. More precisely, in the case where the termination payoff is independent of future uncertainties, increased Knightian uncertainty accelerates the optimal timing policy of an uncertainty averse investor by shrinking the continuation region where exercising is suboptimal. The reason for this result is that in that case higher Knightian uncertainty decreases the value of the optimal policy while it simultaneously leaves the termination payoff unchanged. Since exercising the opportunity is suboptimal as long as the value of the opportunity dominates the termination payoff, we find that increased Knightian uncertainty unambiguously accelerates rational exercise in that case. However, if the termination payoff depends on the evolution of the state variable after exercise as well, then the impact of higher Knightian uncertainty on the optimal timing policy is ambiguous. In that case increased Knightian uncertainty decreases the value of the optimal policy. However, since it simultaneously decreases the expected payoff as well, the net impact of higher Knightian uncertainty is ambiguous and depends on its relative impact on the value of waiting and the termination payoff (cf. Miao and Wang (2007)).

The contents of this study are as follows. In section two we specify the underlying ambiguous stochastic dynamics and introduce the considered optimal timing problem. In section three we then investigate the impact of Knightian uncertainty on standard threshold policies and their values. The main findings on the optimal timing policy of an uncertainty averse and risk neutral investor are then presented in section four. Our general findings are then explicitly illustrated in section five in a model based on geometric Brownian motion. Finally, section six concludes our study.

2 The Model

We assume that the underlying state variable evolves according to a linear, time homogeneous, and regular diffusion process defined on the complete filtered probability space $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ and evolving on the state-space $\mathcal{I} = (a, b) \subseteq \mathbb{R}$ according to the dynamics described by the Itô-stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x, \tag{2.1}$$

where B_t denotes standard $\mathbb{P}$-Brownian motion. We assume that the drift coefficient $\mu : \mathcal{I} \mapsto \mathbb{R}$ and the diffusion coefficient $\sigma : \mathcal{I} \mapsto \mathbb{R}_+$ are locally Lipschitz-continuous functions satisfying the standard growth condition and that $\sigma(x) > 0$ for all $x \in \mathcal{I}$. These assumptions guarantee the existence and uniqueness of a strong solution for the stochastic differential equation (2.1) (cf. Karatzas and Shreve (1991), Theorem 2.9 on p. 289). Finally, we also assume that the boundaries of the state-space of the underlying diffusion are natural and, therefore, that the underlying state variable X cannot reach its boundaries in finite time (for a characterization of the boundary behavior of diffusions, see Borodin and Salminen (2002) pp. 14–21).

In order to characterize the ambiguity faced by the uncertainty averse decision maker, we now follow Chen and Epstein (2002) (see also Nishimura and Ozaki (2007)) and introduce the parameterized family of positive exponential martingales (*Wald's martingale*)

$$Z_t^\theta = e^{-\theta B_t - \frac{1}{2}\theta^2 t},$$

where the parameter $\theta \in [-\kappa, \kappa]$ and $\kappa \in \mathbb{R}_+$ is a constant measuring the degree of Knightian uncertainty (known also as κ-ignorance; cf. Chen and Epstein (2002)). Given the exponential martingale Z_t^θ we can now define the parameterized family of equivalent measures $\mathbb{Q}^\theta$ on $\mathcal{F}_t$ by the likelihood-ratio $d\mathbb{Q}^\theta/d\mathbb{P} = Z_t^\theta$. A standard application of the Girsanov theorem then implies that

$$W_t^\theta = B_t + \theta t$$

is a $\mathcal{F}_t$-martingale with respect to the equivalent measure $\mathbb{Q}^\theta$ (cf. Theorem 8.6.4 in Øksendal (2003) and Borodin and Salminen (2002) pp. 48–50). Hence, for any given fixed θ and equivalent measure $\mathbb{Q}^\theta$ we have

$$dX_t = (\mu(X_t) - \theta\sigma(X_t))dt + \sigma(X_t)dW_t^\theta, \quad X_0 = x. \tag{2.2}$$

As usually, we denote the differential operator associated to the underlying diffusion characterized by the stochastic differential equation (2.2) as

$$\mathcal{A}_\theta = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + (\mu(x) - \theta\sigma(x))\frac{d}{dx}.$$

It is well-known from the classical literature on linear diffusions that there are two linearly independent *fundamental solutions* $\psi_\theta(x)$ and $\varphi_\theta(x)$ satisfying a set of appropriate boundary conditions based on the boundary behavior of the underlying process X and spanning the set of solutions of the ordinary differential equation $(\mathcal{A}_\theta u)(x) = ru(x)$, where $r > 0$ denotes the prevailing constant discount rate (see Borodin and Salminen (2002), pp. $18 - 19$, for a comprehensive characterization of these fundamental solutions). These fundamental solutions constitute the minimal r-excessive mappings for the underlying diffusion. Since any non-trivial r-excessive function can be expressed as a linear combination of these functions (see Borodin and Salminen (2002), p. 33, for a precise characterization) and the value of the optimal timing policy is r-excessive, we observe that the fundamental solutions play a key role in the analysis of the optimal timing policy and its value. Moreover, $\psi_\theta'(x)\varphi_\theta(x) - \varphi_\theta'(x)\psi_\theta(x) = B_\theta S_\theta'(x)$, where $B_\theta > 0$ denotes the constant Wronskian of the fundamental solutions and

$$S_\theta'(x) = \exp\left(-\int \frac{2(\mu(x) - \theta\sigma(x))dx}{\sigma^2(x)}\right)$$

denotes the density of the scale function of X. Following Nishimura and Ozaki (2007) we assume throughout this study that the relations among the underlying variables are not altered as the horizon tends to infinity and, therefore, that the finite horizon characterization remains valid in the infinite horizon setting as well.

Given the underlying stochastically fluctuating state variable X_t and the parameter $\theta \in [-\kappa, \kappa]$, denote now as $\mathcal{L}_\theta^1$ the class of profit flows having a finite expected cumulative present value under the measure $\mathbb{Q}^\theta$. For any cash flow $\pi \in \mathcal{L}_\theta^1$ we denote its expected cumulative present value as

$$(R_r^\theta \pi)(x) = \mathbb{E}_x^{\mathbb{Q}^\theta} \int_0^\infty e^{-rt}\pi(X_t)dt.$$

As is known from the classical theory of diffusions, the expected cumulative present value $(R_r^\theta \pi)(x)$ of a cash flow $\pi \in \mathcal{L}_\theta^1$ can be re-expressed as (cf. Borodin and Salminen (2002), pp. 17–20 and p.

29)

$$(R_r^\theta \pi)(x) = B_\theta^{-1}\varphi_\theta(x)\int_a^x \psi_\theta(y)\pi(y)m_\theta'(y)dy + B_\theta^{-1}\psi_\theta(x)\int_x^b \varphi_\theta(y)\pi(y)m_\theta'(y)dy, \tag{2.3}$$

where $m_\theta'(y) = 2/(\sigma^2(x)S_\theta'(x))$ denotes the density of the speed measure of the underlying state process X_t under the measure $\mathbb{Q}^\theta$.

Our objective is to analyze how Knightian uncertainty characterized by κ-ignorance affects the optimal timing policies of a *risk neutral and uncertainty averse* investor in the case where the underlying state dynamics are characterized by a continuous diffusion process. To this end, we now plan to investigate the optimal timing problem

$$V(x) = \sup_\tau \inf_{\theta\in[-\kappa,\kappa]} \mathbb{E}_x^{\mathbb{Q}^\theta}\left[\int_0^\tau e^{-rs}\pi(X_s)ds + e^{-r\tau}g(X_\tau)\right], \tag{2.4}$$

where τ is an arbitrary stopping time, $\pi : \mathcal{I} \mapsto \mathbb{R}$ is a known continuous cash flow accrued from continuing operation, and the exercise payoff $g : \mathcal{I} \mapsto \mathbb{R}$ is a known continuous function which is assumed to be bounded from below. It is at this point worth emphasizing that in the optimal timing problem (2.4) both the measure as well as the optimal stopping rule are determined by the investor. More precisely, the parameter θ capturing the degree of ambiguity faced by the uncertainty averse investor is first chosen so as to minimize the expected present value of the exercise payoff at any state. The timing rule is then chosen so as to maximize the value of the timing policy in this worst case scenario.

In order to derive the solution of the optimal timing problem (2.4) and characterize how it is associated to the value of standard timing problems in the absence of ambiguity, we also consider the parameterized family of optimal timing problems

$$V_\theta(x) = \sup_\tau \mathbb{E}_x^{\mathbb{Q}^\theta}\left[\int_0^\tau e^{-rs}\pi(X_s)ds + e^{-r\tau}g(X_\tau)\right]. \tag{2.5}$$

It is clear that in the absence of Knightian uncertainty (i.e. when $\kappa = 0$) the considered problem coincides with a standard optimal investment timing problem defined with respect to the underlying state dynamics characterized by (2.1). As soon as Knightian uncertainty is introduced and the maximum degree of uncertainty satisfies the inequality $\kappa > 0$ there is a continuum of equivalent measures $\mathbb{Q}^\theta$ and, therefore, a continuum of associated optimal timing problems indexed by the parameter $\theta \in [-\kappa, \kappa]$.

3 Ambiguity and Threshold Policies

Before proceeding to the analysis of the general investment timing problem of an uncertainty averse investor, we first investigate how Knightian uncertainty affects the value of ordinary single threshold policies arising typically in the literature focusing on real options and the timing of irreversible investment decisions. We first state the following auxiliary result needed in the proof of our main results and the analysis of the comparative static properties of the optimal timing rules and their values in a broad class of investment timing problems.

Lemma 3.1. *Let $\tau_y = \inf\{t \geq 0 : X_t = y\}$ denote the first hitting time of the underlying state variable X_t to an arbitrary constant threshold $y \in \mathcal{I}$ and assume that $\hat{\theta} \geq \theta$. Then*

$$\mathbb{E}_x^{\mathbb{Q}^\theta}\left[e^{-r\tau_y}\right] = \frac{\psi_\theta(x)}{\psi_\theta(y)} \geq \frac{\psi_{\hat{\theta}}(x)}{\psi_{\hat{\theta}}(y)} = \mathbb{E}_x^{\mathbb{Q}^{\hat{\theta}}}\left[e^{-r\tau_y}\right] \tag{3.1}$$

for all $x \leq y$ and

$$\mathbb{E}_x^{\mathbb{Q}^\theta}\left[e^{-r\tau_y}\right] = \frac{\varphi_\theta(x)}{\varphi_\theta(y)} \leq \frac{\varphi_{\hat{\theta}}(x)}{\varphi_{\hat{\theta}}(y)} = \mathbb{E}_x^{\mathbb{Q}^{\hat{\theta}}}\left[e^{-r\tau_y}\right] \tag{3.2}$$

for all $x \geq y$. Moreover, the logarithmic growth rates of the fundamental solutions $\psi_\theta(x)$ and $\varphi_{\hat{\theta}}(x)$ satisfy the inequalities

$$\frac{\psi'_\theta(x)}{\psi_\theta(x)} \leq \frac{\psi'_{\hat{\theta}}(x)}{\psi_{\hat{\theta}}(x)} \tag{3.3}$$

and

$$\frac{\varphi'_\theta(x)}{\varphi_\theta(x)} \geq \frac{\varphi'_{\hat{\theta}}(x)}{\varphi_{\hat{\theta}}(x)} \tag{3.4}$$

for all $x \in \mathcal{I}$.

Proof. See Appendix A. □

Lemma 3.1 states an unambiguous characterization of how the parameter θ affects the expected present value of a unit of money received at the first date the underlying state variable hits an arbitrary constant threshold (i.e. the current value of a zero coupon bond maturing at the random date τ_y). According to Lemma 3.1 this expected value is monotonic as a function of the parameter θ. This result is important since it proves that the parameterized family of expected present values attain its extreme values (as functions of Knightian uncertainty) on the boundaries of the compact set $[-\kappa, \kappa]$. Since increased Knightian uncertainty constitutes a symmetric enlargement of the set $[-\kappa, \kappa]$, we find that Lemma 3.1 also establishes that the sign of the relationship between increased Knightian uncertainty and the parameterized expected present value is unambiguously negative. A second set of auxiliary results summarizing our main findings on the sensitivity of the expected cumulative present value of a monotone cash flow with respect to changes in the parameter θ is now summarized in the following.

Lemma 3.2. *(A) Assume that the cash flow $\pi \in \mathcal{L}_\theta^1$ is non-decreasing and that $\hat{\theta} \geq \theta$. Then $(R_r^\theta \pi)(x) \geq (R_r^{\hat{\theta}} \pi)(x)$ for all $x \in \mathcal{I}$. Especially, $\inf_{\theta \in [-\kappa,\kappa]} (R_r^\theta \pi)(x) = (R_r^\kappa \pi)(x)$ for all $x \in \mathcal{I}$.*
(B) Assume that the cash flow $\pi \in \mathcal{L}_{\hat{\theta}}^1$ is non-increasing and that $\hat{\theta} \geq \theta$. Then $(R_r^\theta \pi)(x) \leq (R_r^{\hat{\theta}} \pi)(x)$ for all $x \in \mathcal{I}$. Especially, $\inf_{\theta \in [-\kappa,\kappa]} (R_r^\theta \pi)(x) = (R_r^{-\kappa} \pi)(x)$ for all $x \in \mathcal{I}$.

Proof. See Appendix B. □

Lemma 3.2 extends the findings of Lemma 3.1 to the expected cumulative present values of monotonic cash flows. Interestingly, and in line with the findings of Lemma 3.1, Lemma 3.2 shows that the sign of the relationship between the parameter θ and the expected cumulative present value depends on whether the cash flow is increasing or decreasing. In case the cash flow is increasing (decreasing) this relationship is negative (positive). Hence, Lemma 3.2 demonstrates that increased Knightian uncertainty decreases the expected cumulative present value of monotone cash flows.

In light of the results of Lemma 3.1 we now introduce the family of functionals $F_\theta^y : \mathcal{I} \mapsto \mathbb{R}$ and $G_\theta^y : \mathcal{I} \mapsto \mathbb{R}$ indexed by the parameter θ and defined as

$$F_\theta^y(x) = \mathbb{E}_x^{\mathbb{Q}^\theta} \left[\int_0^{\tau_{(a,y)}} e^{-rs} \pi(X_s) ds + e^{-r\tau_{(a,y)}} g(X_{\tau_{(a,y)}}) \right] \tag{3.5}$$

and

$$G_\theta^y(x) = \mathbb{E}_x^{\mathbb{Q}^\theta} \left[\int_0^{\tau_{(y,b)}} e^{-rs} \pi(X_s) ds + e^{-r\tau_{(y,b)}} g(X_{\tau_{(y,b)}}) \right] \tag{3.6}$$

where $\tau_{(z,y)} = \inf\{t \geq 0 : X_t \notin (z, y)\}$ denotes the first exit time of the underlying state variable X from the open interval $(z, y) \subset \mathcal{I}$. It is clear that these functionals measure the expected present value of the cumulative payoff accrued by following a timing policy characterized by the hitting time to a fixed single threshold. Given these definitions, we can now establish the following interesting result characterizing how Knightian uncertainty affects the values and continuation regions of standard threshold policies arising in real option models of irreversible investment.

Theorem 3.3. *Assume that $\hat{\theta} \geq \theta$ and that the cash flow $\pi \in \mathcal{L}_\theta^1$ has a finite expected cumulative present value under the measure $\mathbb{Q}^\theta$ for all $\theta \in [-\kappa, \kappa]$. Then*

$$F_\theta^y(x) = \begin{cases} g(x) & x \geq y \\ (R_r^\theta \pi)(x) + (g(y) - (R_r^\theta \pi)(y))\frac{\psi_\theta(x)}{\psi_\theta(y)} & x < y \end{cases} \tag{3.7}$$

and

$$G_\theta^y(x) = \begin{cases} (R_r^\theta \pi)(x) + (g(y) - (R_r^\theta \pi)(y))\frac{\varphi_\theta(x)}{\varphi_\theta(y)} & x > y \\ g(x) & x \leq y. \end{cases} \tag{3.8}$$

Moreover,

(i) if $F_\theta'(x) \geq 0$ for $x \in (a, y)$, then $F_\theta^y(x) \geq F_{\hat{\theta}}^y(x)$ for all $x \in \mathcal{I}$, $\inf_{\theta \in [-\kappa,\kappa]}\{F_\theta^y(x)\} = F_\kappa^y(x)$, and $\{x \in \mathcal{I} : F_{\hat{\theta}}^y(x) > g(x)\} \subseteq \{x \in \mathcal{I} : F_\theta^y(x) > g(x)\}$

(ii) if $G_\theta'(x) \geq 0$ for $x \in (y, b)$, then $G_\theta^y(x) \geq G_{\hat{\theta}}^y(x)$, $\inf_{\theta \in [-\kappa,\kappa]}\{G_\theta^y(x)\} = G_\kappa^y(x)$, and $\{x \in \mathcal{I} : G_{\hat{\theta}}^y(x) > g(x)\} \subseteq \{x \in \mathcal{I} : G_\theta^y(x) > g(x)\}$

(iii) if $F_\theta'(x) \leq 0$ for $x \in (a, y)$, then $F_\theta^y(x) \leq F_{\hat{\theta}}^y(x)$ for all $x \in \mathcal{I}$, $\inf_{\theta \in [-\kappa,\kappa]}\{F_\theta^y(x)\} = F_{-\kappa}^y(x)$, and $\{x \in \mathcal{I} : F_\theta^y(x) > g(x)\} \subseteq \{x \in \mathcal{I} : F_{\hat{\theta}}^y(x) > g(x)\}$

(iv) if $G_\theta'(x) \leq 0$ for $x \in (y, b)$, then $G_\theta^y(x) \leq G_{\hat{\theta}}^y(x)$, $\inf_{\theta \in [-\kappa,\kappa]}\{G_\theta^y(x)\} = G_{-\kappa}^y(x)$, and $\{x \in \mathcal{I} : G_\theta^y(x) > g(x)\} \subseteq \{x \in \mathcal{I} : G_{\hat{\theta}}^y(x) > g(x)\}$.

Proof. See Appendix C. □

Theorem 3.3 states a set of sufficient conditions under which the impact of increased Knightian uncertainty on both the value as well as the continuation region where the value dominates the exercise payoff can be unambiguously characterized. According to Theorem 3.3, the sign of the relationship between higher Knightian uncertainty and the expected present value of the total payoff (the sum of continuation and termination payoffs) accrued from following a threshold policy depends on the monotonicity of the value as a function of the present state of the underlying stochastically fluctuating state variable which, in turn, depends on the relative sizes of the termination and continuation payoffs. Under the conditions of Theorem 3.3, increased Knightian uncertainty is shown to decrease the value of a threshold policy by decreasing both the expected cumulative present value of the cash flow accrued prior exercise and the expected present value of the termination payoff. Theorem 3.3 also characterizes how higher Knightian uncertainty affects the continuation region where the value of a threshold policy dominates the exercise payoff. Interestingly, under the conditions of our theorem, higher Knightian uncertainty is shown to have a stimulating effect on timing by shrinking the continuation region where the value dominates the exercise payoff. Naturally, this result is based on our assumptions on the relative sizes of the termination and continuation payoffs. As our subsequent explicit illustrations will indicate, if these conditions are not met, then our principal findings may actually be reversed. It is also worth emphasizing that our observations indicate that the sign of the relationship between Knightian uncertainty and the expected values of single threshold policies is essentially based on first order monotonicity properties while the impact of increased risk (measured by volatility) is typically based on the second order convexity properties of the values as functions of the underlying state variable. This observation is of interest since it indicates that the impact of these two types of uncertainties on the value of standard threshold policies may be drastically different.

4 Knightian Uncertainty and Optimal Timing

Up to now, we have only considered the impact of Knightian uncertainty on the value of potentially suboptimal timing strategies which are characterizable as ordinary threshold policies requiring that the underlying state variable is stopped whenever it crosses a predetermined constant threshold. However, due to uncertainty aversion it is not clear wether the optimal timing policies are actually characterizable as ordinary threshold strategies or not. Fortunately, as the analysis of our previous section already indicated, the optimal timing policy of an uncertainty averse investor can be determined in typical cases subject to monotone payoffs from an ordinary optimal stopping problem defined with respect a unique well-defined equivalent measure.

Theorem 4.1. *(A) Assume that the exercise payoff $g(x)$ and the cash flow $\pi(x)$ are non-decreasing and continuous. Then, $V(x) = \inf_{\theta \in [-\kappa,\kappa]} V_\theta(x) = V_\kappa(x)$, that is,*

$$V(x) = V_\kappa(x) = \sup_\tau \mathbb{E}_x^{\mathbb{Q}^\kappa} \left[\int_0^\tau e^{-rs} \pi(X_s) ds + e^{-r\tau} g(X_\tau) \right]. \tag{4.1}$$

(B) Assume that the exercise payoff payoff $g(x)$ and the cash flow $\pi(x)$ are non-increasing and continuous. Then, $V(x) = \inf_{\theta \in [-\kappa,\kappa]} V_\theta(x) = V_{-\kappa}(x)$, that is,

$$V(x) = V_{-\kappa}(x) = \sup_\tau \mathbb{E}_x^{\mathbb{Q}^{-\kappa}} \left[\int_0^\tau e^{-rs} \pi(X_s) ds + e^{-r\tau} g(X_\tau) \right]. \tag{4.2}$$

In both cases (A) and (B) increased Knightian uncertainty decreases the value of the investment opportunity and accelerates rational exercise by shrinking the continuation region where waiting is optimal.

Proof. See Appendix D. □

Theorem 4.1 extends the findings of Theorem 3.3 to the case where the investment timing policy is chosen optimally. Along the lines of Theorem 3.3 we again find that the monotonicity of the value of the optimal timing policy as function of the current state of the underlying state variable is the principal determinant of the sign of the relationship between increased Knightian uncertainty and the value of the optimal policy. More specifically, Theorem 4.1 proves that increased Knightian uncertainty unambiguously decreases the value of the optimal timing policy of an uncertainty averse investor whenever the continuation and termination payoffs are monotone. Within our modeling framework, higher Knightian uncertainty increases uniformly the plausibility of favorable as well as unfavorable scenarios. However, since an uncertainty averse investor bases her timing decision on the worst case scenario, only the unfavorable cases count and, therefore, higher Knightian uncertainty unambiguously decreases the value of the optimal timing policy. Theorem 4.1 also proves that higher Knightian uncertainty unambiguously accelerates optimal timing by shrinking the continuation region where exercising the opportunity is suboptimal. The reason for this observation is that increased Knightian uncertainty decreases the value of the optimal timing policy and leaves the termination payoff unchanged. Since timing is suboptimal as long as the value dominates the termination payoff, we find that increased Knightian uncertainty accelerates timing. An interesting implication of the conclusions of Theorem 4.1 on optimal exit policies of uncertainty averse firms is summarized in the following.

Theorem 4.2. *(A) Assume that* $g(x) \equiv 0$ *and that the cash flow* $\pi \in \mathcal{L}_{\kappa}^{1}$ *is continuous, non-decreasing, and satisfies the limiting conditions* $\lim_{x\downarrow a}\pi(x) < 0 < \lim_{x\uparrow b}\pi(x)$. *Then, the value* (2.4) *of the optimal exit policy of an uncertainty averse investor is*

$$V(x) = (R_r^{\kappa}\pi)(x) - \varphi_{\kappa}(x)\inf_{y\leq x}\left[\frac{(R_r^{\kappa}\pi)(y)}{\varphi_{\kappa}(y)}\right] = \begin{cases}(R_r^{\kappa}\pi)(x) - (R_r^{\kappa}\pi)(x_{\kappa}^{*})\frac{\varphi_{\kappa}(x)}{\varphi_{\kappa}(x_{\kappa}^{*})} & x > x_{\kappa}^{*}\\ 0 & x \leq x_{\kappa}^{*},\end{cases}$$

where the optimal exit threshold $x_{\kappa}^{*} = \operatorname{argmin}\{(R_r^{\kappa}\pi)(x)/\varphi_{\kappa}(x)\} \in \pi^{-1}(\mathbb{R}_{-})$ *is the unique root of the ordinary first order condition* $(R_r^{\kappa}\pi)'(x_{\kappa}^{*})\varphi_{\kappa}(x_{\kappa}^{*}) = (R_r^{\kappa}\pi)(x_{\kappa}^{*})\varphi_{\kappa}'(x_{\kappa}^{*})$.

(B) Assume that $g(x) \equiv 0$ *and that the cash flow* $\pi \in \mathcal{L}_{-\kappa}^{1}$ *is, continuous, non-increasing, and satisfies the limiting conditions* $\lim_{x\downarrow a}\pi(x) > 0 > \lim_{x\uparrow b}\pi(x)$. *Then, the value* (2.4) *of the optimal exit policy of an uncertainty averse investor is*

$$V(x) = (R_r^{-\kappa}\pi)(x) - \psi_{-\kappa}(x)\inf_{y\geq x}\left[\frac{(R_r^{-\kappa}\pi)(y)}{\psi_{-\kappa}(y)}\right] = \begin{cases}(R_r^{-\kappa}\pi)(x) - (R_r^{-\kappa}\pi)(x_{-\kappa}^{*})\frac{\psi_{-\kappa}(x)}{\psi_{-\kappa}(x_{-\kappa}^{*})} & x < x_{-\kappa}^{*}\\ 0 & x \geq x_{-\kappa}^{*},\end{cases}$$

where the optimal entry threshold $x_{-\kappa}^{*} = \operatorname{argmin}\{(R_r^{-\kappa}\pi)(x)/\psi_{-\kappa}(x)\} \in \pi^{-1}(\mathbb{R}_{-})$ *is the unique root of the ordinary first order condition* $(R_r^{-\kappa}\pi)'(x_{-\kappa}^{*})\psi_{-\kappa}(x_{-\kappa}^{*}) = (R_r^{-\kappa}\pi)(x_{-\kappa}^{*})\psi_{-\kappa}'(x_{-\kappa}^{*})$.

(C) Increased Knightian uncertainty accelerates exit by decreasing the value of the optimal exit policy of a uncertainty averse investor.

Proof. See Appendix E. □

Theorem 4.2 characterizes the impact of higher Knightian uncertainty on the optimal exit policy of an uncertainty investor. Along the lines of our Theorem 4.1 we find that increased Knightian uncertainty decreases the value of the optimal timing policy and accelerates optimal exit by decreasing the value and leaving the termination payoff unchanged. This result is interesting since it demonstrates that the impact of Knightian uncertainty is radically different with the impact of increased risk (as measured by volatility).

It is worth noticing that the qualitative conclusions of Theorem 4.1 are valid only when the exercise payoff is independent of the degree of Knightian uncertainty. Even though this assumption is satisfied in cases where the valuation is based on the behavior of the underlying up to the exercise date, it is obviously violated in studies considering optimal entry decisions under uncertainty. The reason for this argument is that in those cases the exercise payoff depends both on the prevailing market conditions as well as on their uncertain behavior after the entry opportunity has been exercised. In the optimal entry case, the considered optimal timing problems can typically be expressed as

$$V(x) = \sup_{\tau} \inf_{\theta \in [-\kappa,\kappa]} \mathbb{E}_x^{\mathbb{Q}^\theta} \int_\tau^\infty e^{-rs}\pi(X_s)ds = \sup_{\tau} \inf_{\theta \in [-\kappa,\kappa]} \mathbb{E}_x^{\mathbb{Q}^\theta} \left[e^{-r\tau}(R_r^\theta \pi)(X_\tau) \right]. \tag{4.3}$$

The conclusions of Theorem 4.1 are partially extended to this optimal entry case in the following.

Theorem 4.3. *(A) Assume that the cash flow $\pi \in \mathcal{L}_\kappa^1$ is continuous, non-decreasing, and satisfies the limiting conditions $\lim_{x\downarrow a}\pi(x) < 0 < \lim_{x\uparrow b}\pi(x)$. Then, the value (4.3) of the optimal entry policy of an uncertainty averse investor is*

$$V(x) = \psi_\kappa(x) \sup_{y \geq x} \left[\frac{(R_r^\kappa \pi)(y)}{\psi_\kappa(y)} \right] = \begin{cases} (R_r^\kappa \pi)(x) & x \geq x_\kappa^* \\ (R_r^\kappa \pi)(x_\kappa^*) \frac{\psi_\kappa(x)}{\psi_\kappa(x_\kappa^*)} & x < x_\kappa^*, \end{cases}$$

where the optimal entry threshold $x_\kappa^* = \text{argmax}\{(R_r^\kappa \pi)(x)/\psi_\kappa(x)\} \in \pi^{-1}(\mathbb{R}_+)$ is the unique root of the ordinary first order condition $(R_r^\kappa \pi)'(x_\kappa^*)\psi_\kappa(x_\kappa^*) = (R_r^\kappa \pi)(x_\kappa^*)\psi_\kappa'(x_\kappa^*)$.
(B) Assume that the cash flow $\pi \in \mathcal{L}_{-\kappa}^1$ is, continuous, non-increasing, and satisfies the limiting conditions $\lim_{x\downarrow a}\pi(x) > 0 > \lim_{x\uparrow b}\pi(x)$. Then, the value (4.3) of the optimal entry policy of the uncertainty averse investor is

$$V(x) = \varphi_{-\kappa}(x) \sup_{y \leq x} \left[\frac{(R_r^{-\kappa} \pi)(y)}{\varphi_{-\kappa}(y)} \right] = \begin{cases} (R_r^{-\kappa} \pi)(x_{-\kappa}^*) \frac{\varphi_{-\kappa}(x)}{\varphi_{-\kappa}(x_{-\kappa}^*)} & x > x_{-\kappa}^* \\ (R_r^{-\kappa} \pi)(x) & x \leq x_{-\kappa}^*, \end{cases}$$

where the optimal entry threshold $x_{-\kappa}^* = \text{argmax}\{(R_r^{-\kappa} \pi)(x)/\varphi_{-\kappa}(x)\} \in \pi^{-1}(\mathbb{R}_+)$ is the unique root of the ordinary first order condition $(R_r^{-\kappa} \pi)'(x_{-\kappa}^*)\varphi_{-\kappa}(x_{-\kappa}^*) = (R_r^{-\kappa} \pi)(x_{-\kappa}^*)\varphi_{-\kappa}'(x_{-\kappa}^*)$.
(C) Increased Knightian uncertainty decreases the value of the optimal entry policy of an uncertainty averse investor.

Proof. See Appendix F. □

Theorem 4.3 characterizes the optimal entry policy and its value in the presence of Knightian uncertainty. Along the lines of the findings of Theorem 4.1 we find that the monotonicity of the cash flow is again the principal factor determining the sensitivity of the value of the optimal timing policy with respect to changes in the degree of Knightian uncertainty. However, in contrast with the findings of Theorem 4.1, we now observe that the impact of increased Knightian uncertainty on the optimal entry threshold cannot be typically characterized unambiguously. As was already argued before our theorem, the reason for this finding is that in the present case Knightian uncertainty affects directly both the value of the optimal entry policy as well as the termination payoff measuring the expected cumulative present value of the cash flow accrued from the entry date up to an arbitrarily distant future. Since the continuation region where exercising the opportunity is suboptimal reads as $\{x \in \mathcal{I} : V_\theta(x) > (R_r^\theta \pi)(x)\}$, we find that the sign of the relationship between increased Knightian uncertainty and the optimal threshold depends on the relative impact on these two factors.

It is worth emphasizing that our observations on the impact of increased Knightian uncertainty on the optimal timing of exercise are in line with the findings by Miao and Wang (2007) obtained in a discrete time setting. More precisely, Miao and Wang (2007) established in their study that ambiguity may accelerate or decelerate option exercise depending on its relative impact on the continuation and on the termination payoffs. As our results indicate, the same conclusion is partially valid in the continuous time setting as well. Unfortunately, establishing an unambiguous characterization of the impact of increased Knightian uncertainty on the optimal timing of entry is in the present setting a demanding task. However, as our subsequent explicit illustrations will establish, the qualitative conclusions by Miao and Wang (2007) are satisfied in the case where the underlying dynamics follow geometric Brownian motion.

Having established an unambiguous characterization of the impact of increased Knightian uncertainty on the value of optimal timing policies in the presence of uncertainty aversion, it would naturally be of interest to present such an unambiguous characterization for the impact of increased volatility (or risk) as well. Unfortunately, this is generally impossible since in the presence of Knightian uncertainty increased volatility affects the rate at which the underlying state variable is expected to grow as well. We will illustrate this observation in an explicitly parameterized example based on geometric Brownian motion in the following section.

5 Illustration: Geometric Brownian Motion

In order to illustrate our general results explicitly, we first assume that the underlying stochastic dynamics follow an ordinary geometric Brownian motion characterized under the equivalent measure $\mathbb{Q}^\theta$ by the stochastic differential equation

$$dX_t = (\mu - \theta\sigma)X_t dt + \sigma X_t dB_t^\theta, \quad X_0 = x. \tag{5.1}$$

It is well-known that in this case the fundamental solutions read as $\psi_\theta(x) = x^{\lambda_\theta}$ and $\varphi_\theta(x) = x^{\nu_\theta}$, where

$$\lambda_\theta = \frac{1}{2} - \frac{\mu - \theta\sigma}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu - \theta\sigma}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

denotes the positive and

$$\nu_\theta = \frac{1}{2} - \frac{\mu - \theta\sigma}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu - \theta\sigma}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

denotes the negative root of the characteristic equation $\sigma^2 w(w-1) + 2(\mu - \theta\sigma)w - 2r = 0$. Standard differentiation of the characteristic equation shows that

$$\frac{\partial \lambda_\theta}{\partial \theta} = \frac{2\lambda_\theta}{\sigma(\lambda_\theta - \nu_\theta)} > 0,$$

$$\frac{\partial \nu_\theta}{\partial \theta} = -\frac{2\nu_\theta}{\sigma(\lambda_\theta - \nu_\theta)} > 0,$$

$$\frac{\partial \lambda_\theta}{\partial \sigma} = \frac{2\lambda_\theta(\sigma + \theta - \sigma\lambda_\theta)}{\sigma^2(\lambda_\theta - \nu_\theta)} \gtreqqless 0, \quad \lambda_\theta \lesseqqgtr \frac{\sigma + \theta}{\sigma},$$

and

$$\frac{\partial \nu_\theta}{\partial \sigma} = -\frac{2\nu_\theta(\sigma + \theta - \sigma\nu_\theta)}{\sigma^2(\lambda_\theta - \nu_\theta)} \gtreqqless 0, \quad \nu_\theta \lesseqqgtr \frac{\sigma + \theta}{\sigma}.$$

To illustrate our general findings on the comparative static properties of the optimal timing policy and its value, we now consider four different examples arising in the literature on real options. In order to illustrate the dependence of the impact of increased Knightian uncertainty on the nature of the continuation and termination payoffs, we first consider the valuation and optimal exercise policy of a standard investment opportunity. In that case the exercise payoff reads as $g(x) =$

$(x-K)^+$, where $K>0$ is a known exogenously determined sunk cost. Given the monotonicity and independence of the prevailing degree of Knightian uncertainty of the lump-sum exercise payoff, we notice that if the condition $r>\mu-\sigma\kappa$, guaranteeing that $\psi_\kappa>1$, is satisfied then the value of the optimal investment strategy of an uncertainty averse investor reads as

$$V(x)=V_\kappa(x)=\begin{cases} x-K & x\geq x_\kappa^* \\ \frac{K}{\lambda_\kappa-1}\left(\frac{x}{x_\kappa^*}\right)^{\lambda_\kappa} & x<x_\kappa^* \end{cases}$$

where the optimal exercise boundary is

$$x_\kappa^*=\frac{\lambda_\kappa K}{\lambda_\kappa-1}\in(K,\infty).$$

In light of our results on the sensitivity of the root λ_κ with respect to changes in the degree of Knightian uncertainty we observe that increased Knightian uncertainty accelerates rational exercise by decreasing the optimal investment threshold x_κ^*. The impact of higher volatility is naturally ambiguous due to the non-monotonicity of the root λ_κ as a function of the volatility coefficient σ.

As a second example, consider the case where the exercise payoff reads as $g(x)=(K-x)^+$ (a perpetual put). In contrast to our previous example, the exercise payoff is now non-increasing. It is well-known that as long as the absence of speculative bubbles condition $r>\mu+\kappa\sigma$ is satisfied, the value of the investment opportunity reads as

$$V_{-\kappa}(x)=\begin{cases} (K/(1-\nu_{-\kappa}))\left(\frac{x}{x_{-\kappa}^*}\right)^{\nu_{-\kappa}} & x>x_{-\kappa}^* \\ K-x & x\leq x_{-\kappa}^* \end{cases}$$

where the optimal exercise threshold reads as

$$x_{-\kappa}^*=\frac{\nu_{-\kappa}}{\nu_{-\kappa}-1}K\in(0,K).$$

Given the monotonicity of the root $\nu_{-\kappa}$ as a function of the degree of Knightian uncertainty we notice in accordance with the findings of our Theorem 4.1 that increased Knightian uncertainty accelerates investment timing by increasing the threshold $x_{-\kappa}^*$ and decreasing the value $V_{-\kappa}(x)$ of the investment opportunity.

In order to analyze the effects of Knightian uncertainty affects the continuation payoff as well, consider now the optimal entry strategy of an uncertainty averse investor (cf. Nishimura and Ozaki

(2007)). Since

$$\mathbb{E}_x^{\mathbb{Q}^\kappa} \int_0^\infty e^{-rs} X_s ds = \frac{x}{r-\mu+\sigma\kappa}$$

whenever the absence of speculative bubbles condition $r > \mu - \sigma\kappa$ is met, we find that in this case the exercise payoff reads as

$$g(x) = \frac{x}{r-\mu+\sigma\kappa} - K.$$

Applying a similar approach than in the former example proves that the value of the optimal entry strategy of an uncertainty averse investor reads as

$$V(x) = V_\kappa(x) = \begin{cases} \frac{x}{r-\mu+\sigma\kappa} - K & x \geq \bar{x}_\kappa \\ \frac{K}{\lambda_\kappa - 1}\left(\frac{x}{\bar{x}_\kappa}\right)^{\lambda_\kappa} & x < \bar{x}_\kappa \end{cases}$$

where the optimal entry threshold reads as[1]

$$\bar{x}_\kappa = \left(1 - \frac{1}{\nu_\kappa}\right) rK \in ((r-\mu+\sigma\kappa)K, \infty).$$

As indicated by our Theorem 4.3, the value of the optimal entry policy is a decreasing function of the prevailing degree of Knightian uncertainty κ. Again, the impact of increased volatility on the optimal entry threshold is ambiguous due to the non-monotonicity of the root ν_κ as a function of the volatility coefficient σ. However, since

$$\frac{\partial}{\partial\kappa}\bar{x}_\kappa = \frac{rK}{\nu_\kappa^2}\frac{\partial\nu_\kappa}{\partial\kappa} > 0$$

we find in that higher Knightian uncertainty decelerates optimal entry by increasing the optimal entry threshold at which the opportunity is exercised. This observation is of interest since it indicates that even though the sign of the relationship between ambiguity and the value of the optimal timing policy is determined by the first order monotonicity properties of the exercise payoff, Knightian uncertainty may either decelerate or accelerate optimal investment depending on whether the decision is based on a continuous flow of revenues or a single lump-sum payoff accrued at exercise.

[1]The entry threshold simplifies to the proposed form after noticing that for any $a \in \mathbb{R}$ we have

$$(\lambda_\theta - a)(a - \nu_\theta) = \frac{2}{\sigma^2}\left(r - a(\mu - \sigma\theta) - \frac{1}{2}\sigma^2 a(a-1)\right).$$

Finally, consider as our fourth example the optimal exit problem of an uncertainty averse firm (cf. Alvarez (1998) for an analysis in the standard real option setting). In the present case, the optimal timing problem reads as

$$V(x) = \sup_{\tau} \inf_{\theta\in[-\kappa,\kappa]} \mathbb{E}_x^{\mathbb{Q}^\theta} \int_0^\tau e^{-rs}(X_s - c)ds,$$

where $c > 0$ denotes a flow of constant costs associated to continuing operation. Along the lines of our findings on the optimal entry problem we find that whenever the absence of speculative bubbles condition $r > \mu - \sigma\kappa$ is met, the considered optimal exit problem reads

$$V(x) = \frac{x}{r-\mu+\sigma\kappa} - \frac{c}{r} + \sup_{\tau} \mathbb{E}_x^{\mathbb{Q}^\kappa}\left[e^{-r\tau}\left(\frac{c}{r} - \frac{X_\tau}{r-\mu+\sigma\kappa}\right)\right].$$

It is now clear in light of our general results that in this case the value of the optimal exit strategy reads as

$$V(x) = \begin{cases} \frac{x}{r-\mu+\sigma\kappa} - \frac{c}{r} + \frac{c}{r(1-\nu_\kappa)}\left(\frac{x}{\bar{x}_\kappa}\right)^{\nu_\kappa} & x > \bar{x}_\kappa \\ 0 & x \leq \bar{x}_\kappa, \end{cases}$$

where the optimal exit threshold reads as

$$\bar{x}_\kappa = \left(1 - \frac{1}{\lambda_\kappa}\right) c \in (0, c).$$

As was established in our Theorem 4.3 increased Knightian uncertainty decreases the value of the exit opportunity. However, since

$$\frac{\partial}{\partial\kappa}\bar{x}_\kappa = \frac{c}{\lambda_\kappa^2}\frac{\partial\lambda_\kappa}{\partial\kappa} > 0$$

we find that increased Knightian uncertainty accelerates optimal exit of an uncertainty averse investor by increasing the threshold at which exit is optimal. This observation again emphasizes the radically different impact increased Knightian uncertainty has on investment incentives depending on the intertemporal properties of the exercise payoff.

6 Concluding Comments

In this paper we considered the impact of Knightian uncertainty on the optimal timing policy of a risk neutral and uncertainty averse investor. According to our general results, increased Knightian uncertainty unambiguously decreases the value of the optimal timing policy in all cases. However, our findings demonstrated that the impact of higher Knightian uncertainty on the optimal timing policy depends on the intertemporal specification of the exercise payoff. If the exercise payoff is of the lump-sum type and independent of the future evolution of the underlying state variable, then increased Knightian uncertainty accelerates optimal timing by shrinking the continuation region where exercising is suboptimal. In the case where the exercise payoff depends on the future evolution of the underlying state variable and thereby on the prevailing degree of Knightian uncertainty, the impact of an increase in Knightian uncertainty on the timing is ambiguous and depends on its relative impact on the value of waiting and on the expected present value of the termination payoff. Whichever of these two counteracting effects dominates dictates the sign of the relationship between higher Knightian uncertainty and optimal timing.

Even though our results are based on a relatively general continuous time description of the underlying state variable, a natural extension of our study would be to extend our analysis to a multidimensional setting by introducing more underlying factors subject to Knightian uncertainty. Such an extension would provide valuable information on the combined impact of potentially complex interactions between various risks and Knightian uncertainty on the optimal timing policy of an uncertainty averse decision maker (cf. Chen and Epstein (2002)). Analogously, in light of our results on the impact of Knightian uncertainty on single investment timing decisions, it would naturally be of interest to study how Knightian uncertainty affects the irreversible accumulation policies and, especially, the value of growth and divestment options of uncertainty averse investors. A third natural economically interesting extension would be to consider the impact of ambiguity on the optimal risk adoption policies of uncertainty averse investors (cf. Alvarez and Stenbacka (2004)). In that case both the continuation payoff as well as the termination payoff depend on both the prevailing and on the future uncertainties. Thus, as our present analysis indicates, the impact of Knightian uncertainty on the optimal timing policy and its value may be under such circumstances a non-monotonic function of the prevailing degree of Knightian uncertainty. Wether this conclusion is correct or not is an interesting research topic left for future research.

References

Alvarez, L. H. R. *Exit strategies and price uncertainty: A Greenian Approach*, 1998, *Journal of Mathematical Economics*, **29**, 43 – 56.

Alvarez, L. H. R. and Stenbacka, R. *Optimal Risk Adoption: A Real Options Approach*, 2004, *Economic Theory*, **23**, 123–148.

Borodin, A. and Salminen, P. *Handbook on Brownian motion - Facts and Formulae*, 2nd edition, 2002, Birkhauser, Basel.

Chen, Z. and Epstein, L. *Ambiguity, Risk, and Asset Returns in Continuous Time*, 2002, *Econometrica*, **70**, 1403–1443.

Dixit, A., Pindyck, R., *Investment under uncertainty*, 1994, Princeton UP.

Ellsberg, D. *Risk, Ambiguity, and the Savage Axioms*, 1961, *Quarterly Journal of Economics*, **75**, 643–669.

Epstein, L. *A Definition of Uncertainty Aversion*, 1999, *The Review of Economic Studies*, **66**, 579–608.

Epstein, L. and Wang, T. *Intertemporal Asset Pricing under Knightian Uncertainty*, 1994, *Econometrica*, **62**, 283–322.

Epstein, L. and Schneider, M. *Recursive Multiple-priors*, 2003, *Journal of Economic Theory*, **113**, 1–13.

Ghirardato, P. and Marinacci, M. *Ambiguity Made Presice: A Comparative Foundation*, 2002, *Journal of Economic Theory*, **102**, 251–289.

Gilboa, I. and Schmeidler, D. *Maxmin Expected Utility with Nonunique Prior*, 1989, *Journal of Mathematical Economics*, **18**, 141–153.

Karatzas, I. and Shreve, S. *Brownian Motion and Stochastic Calculus*, 1991, Springer, New York.

Knight, F. *Risk, Uncertainty, and Profit*, 1921, Houghton Miffin, Boston, USA.

Maccheroni, F., Marinacci, M., and Rustichini, A. *Ambiguity Aversion, Robustness, and the Variational Representation of Preferences*, 2006, *Econometrica*, **74**, 1447–1498.

Miao, J. and Wang, N. *Risk, Uncertainty, and Option Exercise*, 2007, Working Paper.

Nishimura, K. G. and Ozaki, H. *Search and Knightian Uncertainty*, 2004, *Journal of Economic Theory*, **119**, 299–333.

Nishimura, K. G. and Ozaki, H. *Irreversible Investment and Knightian Uncertainty*, 2007, *Journal of Economic Theory*, **136**, 668 — 694.

Øksendal, B. *Stochastic Differential Equations: An Introduction with Applications*, (Sixth Edition) 2003, Springer, Berlin.

A Proof of Lemma 3.1

Proof. Consider first the case where $x \leq y$. It is well-known from the literature on the classical theory of diffusions that (cf. Borodin and Salminen (2002), p. 18)

$$\mathbb{E}_x^{\mathbb{Q}^\theta}\left[e^{-r\tau_y}\right] = \frac{\psi_\theta(x)}{\psi_\theta(y)},$$

where $\psi_\theta(x)$ the increasing fundamental solution of the ordinary differential equation $(\mathcal{A}_\theta u)(x) = ru(x)$. Since $\theta \leq \hat{\theta}$ we observe that the monotonicity of $\psi_\theta(x)$ and the positivity of the diffusion coefficient $\sigma(x)$ implies that

$$(\mathcal{A}_{\hat{\theta}}\psi_\theta)(x) - r\psi_\theta(x) = ((\mathcal{A}_{\hat{\theta}} - \mathcal{A}_\theta)\psi_\theta)(x) = (\theta - \hat{\theta})\sigma(x)\psi_\theta'(x) \leq 0$$

for all $x \in \mathcal{I}$. Thus, the continuity of the diffusion X_t and Dynkin's theorem now implies that

$$\psi_\theta(x) \geq \mathbb{E}_x^{\mathbb{Q}^{\hat{\theta}}}\left[e^{-r\tau_y}\psi_\theta(X_{\tau_y})\right] = \psi_\theta(y)\mathbb{E}_x^{\mathbb{Q}^{\hat{\theta}}}\left[e^{-r\tau_y}\right] = \psi_\theta(y)\frac{\psi_{\hat{\theta}}(x)}{\psi_{\hat{\theta}}(y)}$$

proving inequality (3.1). Establishing inequality (3.2) is now entirely analogous. It remains to establish inequalities (3.3) and (3.4). To accomplish this task we notice that inequality (3.1) implies that for all $x \leq y$ we have

$$1 - \frac{\psi_{\hat{\theta}}(x)}{\psi_{\hat{\theta}}(y)} \geq 1 - \frac{\psi_\theta(x)}{\psi_\theta(y)}$$

which can be equivalently written as

$$\int_x^y \frac{\psi_{\hat{\theta}}'(s)}{\psi_{\hat{\theta}}(y)}ds \geq \int_x^y \frac{\psi_\theta'(s)}{\psi_\theta(y)}ds.$$

Applying the mean value theorem of integral calculus and letting $x \uparrow y$ proves (3.3). Establishing inequality (3.4) is analogous. □

B Proof of Lemma 3.2

Proof. (A) It is known from the literature on linear diffusions that for $\pi \in \mathcal{L}_\theta^1$ the expected cumulative present value $(R_r^\theta \pi)(x)$ constitutes the unique bounded solution of the ordinary differential equation $(\mathcal{A}_\theta R_r^\theta \pi)(x) - r(R_r^\theta \pi)(x) + \pi(x) = 0$. In light of this identity we find that

$$(\mathcal{A}_{\hat{\theta}} R_r^\theta \pi)(x) - r(R_r^\theta \pi)(x) + \pi(x) = (\theta - \hat{\theta})\sigma(x)(R_r^\theta \pi)'(x) < 0$$

since the expected cumulative present value preserves the monotonicity of the cash flow, $\sigma(x) > 0$ for $x \in \mathcal{I}$, and $\theta \leq \hat{\theta}$. Consequently, applying Dynkin's theorem to the expected cumulative present value $(R_r^\theta \pi)(x)$ yields

$$\mathbb{E}_x^{\mathbb{Q}^{\hat{\theta}}}\left[e^{-rT_n}(R_r^\theta \pi)(X_{T_n})\right] \leq (R_r^\theta \pi)(x) - \mathbb{E}_x^{\mathbb{Q}^{\hat{\theta}}}\int_0^{T_n} e^{-rs}\pi(X_s)ds$$

where $T_n = n \wedge \inf\{t \geq 0 : X_t \notin (\max(a+1/n, -n), \min(n, b-1/n))\}$, $n \in \mathbb{N}$, is an increasing sequence of almost surely finite stopping times tending towards ∞ as $n \to \infty$. Letting $n \to \infty$ and invoking the boundedness from below of the cash flow $\pi(x)$ then proves that $(R_r^{\hat{\theta}}\pi)(x) \leq (R_r^\theta \pi)(x)$ for all $x \in \mathcal{I}$. Establishing part (B) is analogous. □

C Proof of Theorem 3.3

Proof. Applying the strong Markov property, the time homogeneity, and the continuity of the underlying diffusion shows that for all $x \in (a, y)$ we have

$$\begin{aligned} F_\theta^y(x) &= \mathbb{E}_x^{\mathbb{Q}^\theta}\left[\int_0^{\tau_{(a,y)}} e^{-rs}\pi(X_s)ds + e^{-r\tau_{(a,y)}}g(X_{\tau_{(a,y)}})\right] \\ &= (R_r^\theta \pi)(x) + \mathbb{E}_x^{\mathbb{Q}^\theta}\left[e^{-r\tau_{(a,y)}}\left(g(X_{\tau_{(a,y)}}) - (R_r^\theta \pi)(X_{\tau_{(a,y)}})\right)\right] \\ &= (R_r^\theta \pi)(x) + (g(y) - (R_r^\theta \pi)(y))\mathbb{E}_x^{\mathbb{Q}^\theta}\left[e^{-r\tau_{(a,y)}}\right]. \end{aligned}$$

Noticing that $F_\theta^y(x) = g(x)$ for all $x \in (y, b)$ proves the representation (3.7). Establishing (3.8) is entirely analogous.

It remains to characterize the impact of increased Knightian uncertainty on the value of an ordinary threshold policy and on the continuation region where the value of the threshold policy dominates the termination payoff. Assume first that $F_\theta^{y\prime}(x) \geq 0$ for all $x \in (a, y)$. Since $F_\theta^y(x) = F_{\hat{\theta}}^y(x)$ for all $x \in (y, b)$ it is sufficient to consider the behavior of $F_\theta^y(x)$ on the set (a, y). As is known from the literature on linear diffusions, $F_\theta^y(x)$ satisfies on (a, y) the boundary value problem

$$(\mathcal{A}_\theta F_\theta^y)(x) - rF_\theta^y(x) + \pi(x) = 0, \quad F_\theta^y(y) = g(y).$$

Relying now on the proof of Lemma 3.2 proves that $F_\theta^y(x) \geq F_{\hat{\theta}}^y(x)$ for all $x \in (a, y)$ and, therefore, that $F_\theta^y(x) \geq F_{\hat{\theta}}^y(x)$ for all $x \in \mathcal{I}$. If $\{x \in \mathcal{I} : F_\theta^y(x) > g(x)\} = \emptyset$ then $\{x \in \mathcal{I} : F_{\hat{\theta}}^y(x) > g(x)\} = \emptyset$ as well. Assume, therefore, that $\{x \in \mathcal{I} : F_\theta^y(x) > g(x)\} \neq \emptyset$ and let $x \in \{x \in \mathcal{I} : F_{\hat{\theta}}^y(x) > g(x)\}$.

Since $F^y_\theta(x) \geq F^y_{\hat\theta}(x)$, we find that $x \in \{x \in \mathcal{I} : F^y_\theta(x) > g(x)\}$ as well and, therefore, that $\{x \in \mathcal{I} : F^y_{\hat\theta}(x) > g(x)\} \subseteq \{x \in \mathcal{I} : F^y_\theta(x) > g(x)\}$. Establishing the claims of parts (ii), (iii), and (iv) is completely analogous. □

D Proof of Theorem 4.1

Proof. Define the functional $\mathcal{U}^T_\theta : [0,T] \times \mathcal{I} \mapsto \mathbb{R}$ as

$$\mathcal{U}^T_\theta(t,x) = \mathbb{E}^{\mathbb{Q}^\theta}_{(t,x)}\left[\int_t^T e^{-r(s-t)}\pi(X_s)ds + e^{-r(T-t)}g(X_T)\right] \tag{D.1}$$

where $g : \mathcal{I} \mapsto \mathbb{R}$ is the known continuous payoff received at exercise and $\pi(x)$ is the cash flow which is accumulated prior exercise. Under our assumptions $\mathcal{U}^T_\theta(t,x)$ constitutes for $(t,x) \in [0,T] \times \mathcal{I}$ the solution of the boundary value problem (cf. Øksendal (2003), Theorem 9.3.3.)

$$\begin{aligned}(\mathcal{G}^\theta_r \mathcal{U}^T_\theta)(t,x) + \frac{\partial}{\partial t}\mathcal{U}^T_\theta(t,x) + \pi(x) &= 0\\ \mathcal{U}^T_\theta(T,x) &= g(x),\end{aligned}$$

where the differential operator $\mathcal{G}^\theta_r$ is defined for sufficiently smooth mappings $f : \mathcal{I} \mapsto \mathbb{R}$ by $(\mathcal{G}^\theta_r f)(x) = (\mathcal{A}_\theta f)(x) - rf(x)$. Standard differentiation now shows that for all $\hat\theta > \theta$ and $(t,x) \in [0,T] \times \mathcal{I}$ we have

$$(\mathcal{G}^\theta_r \mathcal{U}^T_{\hat\theta})(t,x) + \frac{\partial}{\partial t}\mathcal{U}^T_{\hat\theta}(t,x) + \pi(x) = (\hat\theta - \theta)\sigma(x)\frac{\partial}{\partial x}\mathcal{U}^T_{\hat\theta}(t,x).$$

In light of this result we find by applying Itô's theorem to the mapping $(t,x) \mapsto e^{-rt}\mathcal{U}^T_{\hat\theta}(t,x)$ that

$$\mathbb{E}^{\mathbb{Q}^\theta}_{(t,x)}\left[e^{-rT}g(X_T)\right] = e^{-rt}\mathcal{U}^T_{\hat\theta}(t,x) + \mathbb{E}^{\mathbb{Q}^\theta}_{(t,x)}\int_t^T e^{-rs}(\hat\theta - \theta)\sigma(X_s)\frac{\partial}{\partial x}\mathcal{U}^T_{\hat\theta}(s,X_s)ds$$

Since the expected value $\mathcal{U}^T_\theta(t,x)$ preserves the monotonicity of the exercise payoff $g(x)$ and the cash flow $\pi(x)$, $\hat\theta \geq \theta$, and $\sigma(x) > 0$ for all $x \in \mathcal{I}$ we find that

$$\mathbb{E}^{\mathbb{Q}^\theta}_{(t,x)}\left[\int_t^T e^{-r(s-t)}\pi(X_s)ds + e^{-r(T-t)}g(X_T)\right] \geq \mathbb{E}^{\mathbb{Q}^{\hat\theta}}_{(t,x)}\left[\int_t^T e^{-r(s-t)}\pi(X_s)ds + e^{-r(T-t)}g(X_T)\right]$$

for all $(t,x) \in [0,T] \times \mathcal{I}$ whenever the exercise payoff $g(x)$ and the cash flow $\pi(x)$ are non-decreasing and that the opposite inequality is valid whenever the exercise payoff $g(x)$ and the cash flow $\pi(x)$ are non-increasing. Consequently, we find that

$$\inf_{\theta \in [-\kappa,\kappa]} \mathbb{E}^{\mathbb{Q}^\theta}_{(t,x)}\left[\int_t^T e^{-r(s-t)}\pi(X_s)ds + e^{-r(T-t)}g(X_T)\right] = \mathcal{U}^T_\kappa(t,x)$$

in the former case and

$$\inf_{\theta\in[-\kappa,\kappa]}\mathbb{E}^{\mathbb{Q}^\theta}_{(t,x)}\left[\int_t^T e^{-r(s-t)}\pi(X_s)ds+e^{-r(T-t)}g(X_T)\right]=\mathcal{U}^T_{-\kappa}(t,x)$$

in the latter case.

In light of these inequalities, assume now that both the exercise payoff $g(x)$ and the cash flow $\pi(x)$ are non-decreasing and consider the increasing sequence of functions $\{v_n(x)\}_{n\in\mathbb{Z}_+}$ defined iteratively as $v_0(x)=g(x)$ and

$$\begin{aligned}v_{k+1}(x) &= \sup_{T\geq 0}\inf_{\theta\in[-\kappa,\kappa]}\mathbb{E}^{\mathbb{Q}^\theta}_x\left[\int_0^T e^{-rs}\pi(X_s)ds+e^{-rT}v_k(X_T)\right]\\ &= \sup_{T\geq 0}\mathbb{E}^{\mathbb{Q}^\kappa}_x\left[\int_0^T e^{-rs}\pi(X_s)ds+e^{-rT}v_k(X_T)\right]\end{aligned}$$

for $k\geq 1$. As is known from the literature on optimal stopping the sequence $v_n(x)$ converges towards the value of the optimal stopping problem (2.4) and, therefore, that

$$V(x)=\sup_{\tau}\mathbb{E}^{\mathbb{Q}^\kappa}_x\left[\int_0^\tau e^{-rs}\pi(X_s)ds+e^{-r\tau}g(X_\tau)\right]$$

proving our first claim. Establishing the alleged claim in the case where the exercise payoff $g(x)$ and the cash flow $\pi(x)$ are non-increasing is completely analogous. The alleged sensitivities of the value of the optimal policy and the continuation region where exercising the opportunity is suboptimal now follow from Theorem 3.3. □

E Proof of Theorem 4.2

Proof. (A) We consider the optimal exit problem for an arbitrary θ since the monotonicity of the solution as a function of the parameter θ is then sufficient for deriving the optimal policy of an uncertainty averse investor. We first notice that applying the Green-representation (2.3) to the functional

$$(L_{\varphi_\theta}(R^\theta_r\pi))(x)=\frac{(R^\theta_r\pi)'(x)}{S'_\theta(x)}\varphi_\theta(x)-\frac{\varphi'_\theta(x)}{S'_\theta(x)}(R^\theta_r\pi)(x)$$

yields

$$(L_{\varphi_\theta}(R^\theta_r\pi))(x)=\int_x^b\varphi_\theta(y)\pi(y)m'_\theta(y)dy.$$

Since $\pi(x) > 0$ for all $x \in (\pi^{-1}(0), \infty)$ we observe that $(L_{\varphi_\theta}(R_r^\theta \pi))(x) > 0$ for all $x \in (\pi^{-1}(0), \infty)$. Assume now that $k < \pi^{-1}(0)$. The monotonicity of the cash flow $\pi(x)$ then implies that for all $x < k$ we have

$$(L_{\varphi_\theta}(R_r^\theta \pi))(x) \leq \frac{\pi(k)}{r}\left[\frac{\varphi_\theta'(k)}{S_\theta'(k)} - \frac{\varphi_\theta'(x)}{S_\theta'(x)}\right] + (L_{\varphi_\theta}(R_r^\theta \pi))(k).$$

Since $\varphi_\theta'(x)/S_\theta'(x) \downarrow -\infty$ as $x \downarrow a$ (a was assumed to be a natural boundary) and $\pi(k) < 0$ we find that equation $(L_{\varphi_\theta}(R_r^\theta \pi))(x) = 0$ has a unique root $x_\kappa^* \in (a, \pi^{-1}(0))$. Noticing that

$$(L_{\varphi_\theta}(R_r^\theta \pi))(x) = \frac{\varphi_\theta^2(x)}{S_\theta'(x)} \frac{d}{dx}\left[\frac{(R_r^\theta \pi)(x)}{\varphi_\theta(x)}\right]$$

shows that $x_\theta^* = \operatorname{argmin}\{(R_r^\theta \pi)(x)/\varphi_\theta(x)\}$. Given this observation it is now straightforward to establish that the proposed value function satisfies the conditions $V_\theta \in C^1(\mathcal{I}) \cap C^2(\mathcal{I} \backslash \{x_\theta^*\})$, $|V_\theta''(x_\theta^* \pm)| < \infty$, $V_\theta(x) \geq 0$ for all $x \in \mathcal{I}$, and $(\mathcal{G}_r^\theta V_\theta)(x) + \pi(x) \leq 0$ for all $x \in \mathcal{I} \backslash \{x_\theta^*\}$. Therefore, it dominates the value of the optimal exit problem (2.5). However, since the proposed value is attained by the admissible threshold policy $\tau_{(x_\theta^*, b)}$ we find that it coincides with the value of the optimal exit problem (2.5) and, therefore, that $\tau_{(x_\theta^*, b)}$ is the optimal exit date. Moreover, noticing that

$$\frac{d}{dx}\left[\frac{(R_r^\theta \pi)'(x)}{\varphi_\theta'(x)}\right] = \frac{2r S_\theta'(x)}{\sigma^2(x) {\varphi_\theta'}^2(x)} \int_x^b \varphi_\theta(y)(\pi(x) - \pi(y)) m_\theta'(y) dy < 0$$

proves that

$$(R_r^\theta \pi)'(x) \geq \varphi_\theta'(x) \frac{(R_r^\theta \pi)'(x_\theta^*)}{\varphi_\theta'(x_\theta^*)}$$

for all $x \geq x_\theta^*$ and, therefore, that the value is nondecreasing. Establishing the claims of part (B) is entirely analogous. Finally, part (C) follows directly from Theorem 4.1. □

F Proof of Theorem 4.3

Proof. (A) Before proceeding in the proof of the alleged results, we first notice that since the termination payoff now reads as $g(x) = (R_r^\theta \pi)(x)$ the auxiliary functional $F_\theta^y(x)$ can now be expressed as

$$F_\theta^y(x) = \mathbb{E}_x^{\mathbb{Q}^\theta}\left[e^{-r\tau_{(a,y)}} (R_r^\theta \pi)(X_{\tau_{(a,y)}})\right] = \begin{cases} (R_r^\theta \pi)(x) & x \geq y \\ (R_r^\theta \pi)(y) \frac{\psi_\theta(x)}{\psi_\theta(y)} & x < y. \end{cases}$$

Hence, applying Lemma 3.1 and Lemma 3.2 proves that $F_\theta^y(x) \geq F_{\hat{\theta}}^y(x)$ for all $x \in \mathcal{I}$, $y \in \mathcal{I}$, and $\hat{\theta} \geq \theta$. Along the lines of our Theorem 3.3, this shows that $\operatorname{argmin}_{\theta \in [-\kappa, \kappa]}\{F_\theta^y(x)\} = \kappa$.

Having established the monotonicity of the functionals $F_\theta^y(x)$ as functions of Knigtian uncertainty, we now derive the value $V_\theta(x)$ under the assumption that $\pi \in \mathcal{L}_\theta^1$. Consider the continuously differentiable functional

$$(L_{\psi_\theta}(R_r^\theta\pi))(x) = \frac{(R_r^\theta\pi)'(x)}{S_\theta'(x)}\psi_\theta(x) - \frac{\psi_\theta'(x)}{S_\theta'(x)}(R_r^\theta\pi)(x).$$

Applying the Green representation (2.3) yields

$$(L_{\psi_\theta}(R_r^\theta\pi))(x) = -\int_a^x \psi_\theta(y)\pi(y)m_\theta'(y)dy$$

which implies that

$$(L_{\psi_\theta}(R_r^\theta\pi))'(x) = -\psi_\theta(x)\pi(x)m_\theta'(x) \gtreqless 0, \quad x \lesseqgtr \pi^{-1}(0).$$

In light of the assumed monotonicity and boundary behavior of the cash flow $\pi(x)$ we observe that $(L_{\psi_\theta}(R_r^\theta\pi))(x) > 0$ for all $x \leq \pi^{-1}(0)$. Let $x > k > \pi^{-1}(0)$. Invoking the additivity of the functional $(L_{\psi_\theta}(R_r^\theta\pi))(x)$ and the monotonicity of the cash flow $\pi(x)$ yields

$$\begin{aligned}(L_{\psi_\theta}(R_r^\theta\pi))(x) &= (L_{\psi_\theta}(R_r^\theta\pi))(k) - \int_k^x \pi(y)\psi_\theta(y)m_\theta'(y)dy \\ &\leq (L_{\psi_\theta}(R_r^\theta\pi))(k) - \frac{\pi(k)}{r}\left[\frac{\psi_\theta'(x)}{S_\theta'(x)} - \frac{\psi_\theta'(k)}{S_\theta'(k)}\right] \downarrow -\infty\end{aligned}$$

as $x \uparrow b$, since $\psi_\theta'(x)/S_\theta'(x) \uparrow \infty$ when b is a natural boundary for the underlying diffusion (cf. Borodin and Salminen (2002), p. 19). Therefore, equation $(L_{\psi_\theta}(R_r^\theta\pi))(x) = 0$ has a unique root on $x_\theta^* \in (\pi^{-1}(0), b)$ by the continuity and monotonicity of $(L_{\psi_\theta}(R_r^\theta\pi))(x)$. Noticing that

$$(L_{\psi_\theta}(R_r^\theta\pi))(x) = \frac{\psi_\theta^2(x)}{S_\theta'(x)}\frac{d}{dx}\left[\frac{(R_r^\theta\pi)(x)}{\psi_\theta(x)}\right]$$

shows that $x_\theta^* = \text{argmax}\{(R_r^\theta\pi)(x)/\psi_\theta(x)\}$ and, therefore, that $F_\theta^{x_\theta^*}(x) \geq (R_r^\theta\pi)(x)$ for all $x \in \mathcal{I}$. Moreover, we find that $F_\theta^{x_\theta^*} \in C^1(\mathcal{I}) \cap C^2(\mathcal{I}\backslash\{x_\theta^*\})$, $|D_x^2F_\theta^{x_\theta^*}(x_\theta^*\pm)| < \infty$, $(\mathcal{A}_\theta F_\theta^{x_\theta^*})(x) - rF_\theta^{x_\theta^*}(x) = 0$ for $x \in (0, x_\theta^*)$ and $(\mathcal{A}_\theta F_\theta^{x_\theta^*})(x) - rF_\theta^{x_\theta^*}(x) = -\pi(x) < 0$ for $x \in (x_\theta^*, b)$. Consequently, $F_\theta^{x_\theta^*}(x)$ constitutes a r-excessive majorant of the exercise payoff $(R_r^\theta\pi)(x)$. Since $V_\theta(x)$ is the smallest of these majorants, we have that $V_\theta(x) \leq F_\theta^{x_\theta^*}(x)$ for all $x \in \mathcal{I}$. In order to establish the opposite inequality, it is sufficient to notice that since

$$F_\theta^{x_\theta^*}(x) = \psi_\theta(x)\sup_{y\geq x}\left\{\frac{(R_r^\theta\pi)(y)}{\psi_\theta(y)}\right\} = \mathbb{E}_x^{\mathbb{Q}_\theta}\left[e^{-r\tau_{(a,x_\theta^*)}}(R_r^\theta\pi)(X_{\tau_{(a,x_\theta^*)}})\right]$$

and $\tau_{(a,x^*_\theta)}$ is an admissible stopping time (being a Markov-time), we have that $V_\theta(x) \geq F_\theta^{x^*_\theta}(x)$ for all $x \in \mathcal{I}$ and, therefore, that $V_\theta(x) = F_\theta^{x^*_\theta}(x)$. Moreover, it is clear from the proof of Theorem 4.1 that in this case $\inf_{\theta\in[-\kappa,\kappa]} V_\theta(x) = V_\kappa(x) = V(x)$. Establishing part (B) is analogous. Finally, the negativity of the sign of the relationship between higher Knightian uncertainty and the value of the optimal timing policy follows directly from Lemma 3.1 and Lemma 3.2. □

Aboa Centre for Economics (ACE) was founded in 1998 by the departments of economics at the Turku School of Economics, Åbo Akademi University and University of Turku. The aim of the Centre is to coordinate research and education related to economics in the three universities.

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