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ABSTRACT

We study a dynamic vNM stable set in a compact metric space under the assumption of complete and continuous dominance relation. Internal and external stability are defined with respect to farsighted dominance. Stability of an outcome is conditioned on the history via which it is reached. A dynamic stable set always exists. Any covering set by Dutta (1988) coincides with the set of outcomes that are implementable via a dynamic stable set. The maximal implementable outcome set is a version of the ultimate uncovered set.

JEL Classification: C71, C72

Keywords: vNM stable set, dynamic, history

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1 Introduction

The classic question in social choice and voting theory concerns how to choose a social alternative under the majority rule. The situation is captured in a compact form by the set of alternatives X and a *dominance relation* on X, reflecting the majority preference relation. There is only little concensus what is the solution to the problem. Ideally, the solution is institution free in a sense that the implemented social alternative depends on on the basic data of the problem and not on the details of the system that delivers it. The best understood solution concept in this vein is the notion of a Condorcet winner, defined as the outcome that is not dominated by any other outcome. When a Condorcet winner exist, it also emerges as an equilibrium outcome in a large class of electoral institutions.[1]

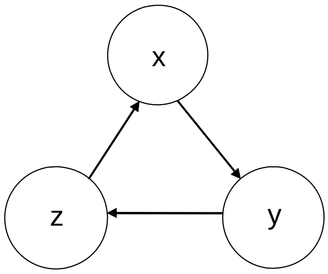
Unfortunately, a Condorcet winner frequently fails to exist. What is more, the so called chaos theorem (e.g. McKelvey 1986) shows that under very general conditions, majority rule exhibits global cycling. Hence any model that explains oucome formation under the majjority rule must have a different take on the stability of the system. Social choice theory has been remarkably succesful in developing and analysing solutions from axiomatic groundings (Laslier 1997 is an thorough survey). However, these solutions usually lack a convincing dynamic story: how would the implemented outcome eventually be reached and why can the parties commit to implementing it? This is since the models deliberately abstract from strategic considerations.

Von Neumann -Morgenstern (vNM) (1944) stable set is an exception. A stable set $S \subseteq X$ is defined by the internal and external stability concepts: (i) S meets internal stability if, for all $x \in S$ there is no $y \in S$ such that x dominates y, and (ii) S meets internal stability if, for all $x \notin S$, there is $y \in S$ such that x dominates y. These conditions can be taken to imply that the model free of inner strategic contradictions as no majority coalition benefits from moving from a stable outcome to another stable outcome. Moreover, it explains why a majority coaltion always wants to move the play from an unstable outcome to a stable one. Important recent contributions examining the scope of stable sets include Dutta (1988), Chwe (1994), and Xue (1998). Greenberg (1990) provides a useful taxonomy of stable set -approaches.

However also the stable set suffers from the existence problem. Consider a three-cycle on $X = \{x, y, z\}$ such that x dominates y, y dominates z, and z dominates x. If $S \subseteq X$ is a stable set, then, by internal stability, S is a singleton, say $\{x\}$. But then it violates external stability since x does not dominate z (see Fig. 1, where $x \to y$ means x dominates y, etc.).

[1] Including electoral competition (Downs, 1957), representative democracy (Besley and Coate, 1997), and agenda-setting (Ferejohn, Fiorina, McKelvey, 1987).

The argument of this paper is that the existence problem of the stable set solution is due to stationarity of the solution, i.e. its reliance on structures that do allow history dependency.



Harsanyi (1974) criticized the stable set solution on the grounds that it fails to take farsightedness into account appropriately (see also Chwe, 1994, and Xue, 1998). That is, once a status quo in the stable set is challenged by an element outside the stable set, then internal stability only indicates that *some* element inside the set dominates the new status quo, and forces the play move back inside. The criticism is that there is no guarantee that the third outcome is not precisely what the party that upset the original stable outcome wanted, i.e. that the third outcome would not dominate the first one. If this is the case, then that is precisely the reason why the first status quo should not be regarded as stable contradicting the original construction. As a remedy, Harsanyi (1970) suggests an indirect dominance criterion: sequence $(x_0, ..., x_K) \in X^K$ is an *indirect dominance chain* if $x_K R x_k$, for all $k = 0, ..., K-1$. The problem is, however, that *chains* (as opposed to their end-points) are difficult to use as a dominance criterion in the standard, outcome based approach.

The aim of this paper is to define and study a *dynamic* version of the stable set. The stable sets are studied in a tournament context; a tournament R is a complete, asymmetric, irreflexive relation over the set of alternatives X. By dynamicity I mean that the process has a *memory* - what has happened in the past may have consequences in the future - as well as *farsightedness.* More formally, the dynamic stable set V is defined over the set of all finite sequences (histories) of alternatives $H = \{x^*\} \cup_{k=0}^{\infty} X^k$, where x^* is a distinguished *initial* outcome. The dominance relation over H is now extended from the original relation R on X in the following way: an element $(h, x) \in H$ is dominated by $(h, x, ..., y) \in H$ if $(x, ..., y)$ is an indirect dominance chain. The dynamic stable set $V \subseteq H$ is defined with respect to this dominance relation.

My main result is that there always exists a dynamic stable set that is defined with respect to the indirect dominance relation. A complete characterization of the outcomes implementable via such dynamic stable sets are characterized. Since a chain (h, x) in a dynamic stable set is not credibly objected after its final outcome, x, is reached, x can be interpreted as the implemented outcome. My principal object of study is the set of outcomes that are *implementable* via a dynamic stable set, i.e. the set of outcomes x such that $(h, x) \in V$. I show that the set of outcomes that are implementable via a dynamic stable set can be characterized by a solution called a *consistent choice set*. A consistent choice set $C \subseteq X$ satisfies the following feature: if an element in C is dominated by another element, then there is an element in C that forms a three-cycle with the two former elements.[2] The existence of a covering set - which is not clear a priori - is proven by showing that any *covering set* by Dutta (1988) is a consistent choice set. Since the former is known to exist, e.g. the *ultimate uncovered set* of the tournament R is a covering set, also a consistent choice set exists.[3] Hence, *a fortiori*, a dynamic stable set exists. Finally, I show that the ultimate uncovered set is the unique *maximal* consistent choice set.[4]

In more concrete terms, think of endogenous agenda formation under the majority criterion. The rules of the agenda setting game are simple: An outcome herited from the history is on the table. Then any majority coalition may challenge the outcome on the table by proposing (any) another outcome. The new outcome then becomes the alternative on the table. When the alternative on the table is not challenged, it is implemented. But after the new outcome has become the alternative on the table, a new majority may intervene and replace it with a yet third one, and so forth... . When to stop? The standard *stationary* view of stable set cannot answer since history independent coalitional strategies force a coalitions do not allow coalitions to *punish* other coalitions that have deviated from a stable standard. Without a flexible punishment rule, consistent and rational behavior cannot be entertained (see Barberà and Gerber, 2007). The contribution of this paper is to show that if history dependent coalitional strategies are permitted, then a solution does exist.

Agenda formation has been an important topic in the literature. The distinctive feature of our approach is the unboundedness of the agenda formation process. Most of the literature assume a fixed, finitely long agenda. E.g. Moulin (1979), Shepsle and Weingast, (1984), and Banks (1985) analyze majority voting in finite elimination trees. Dutta, Jackson and Le Breton (2001a, see also 2001b, 2002) study endogenous agenda formation with bounded maximum length of the resulting agenda. With unbounded agenda formation process, backwards induction cannot be used to solve the model. It needs to be identified via internal consistency considerations

Example To see how a dynamic stable set works, consider again the above 3-cycle. Partition the set of histories $H = \cup_k \{x, y, z\}^k$ into two *phases,*

[2] To my best knowledge, the consistent choice set is a new solution.

[3] The uncovered set is due to Fishburn (1977), and Miller (1980). Iteration of the uncovered set is studied e.g. by Dutta (1988), and Coughlan and LeBreton (1999).

[4] For recent contributions on this, see Penn (2006) and Patty and Penn (2007).

P_1 and P_2 (start with, say, P_1). The partition is implicitly defined by transition from the current phase to the other phase whenever the history ending with x is extended by y, i.e. one moves from (h, x) to (h, x, y). Given this implicit partitioning of histories h, each history $(h, x), (h, y)$, and (h, z) can be categorized belonging in either P_1 or in P_2. Denoting by $P_1(x)$ the set of histories $P_1 \cap \{(h, x) : h \in H\}$, and similarly for $P_1(y)$ and $P_2(z)$, construct the stable set to be the union of $P_1(x)$, $P_1(y)$, and $P_2(x)$. In Fig. 2, the dynamic stable set is depicted as the shaded nodes. Because of the cyclicity the dominance relation, all indirect dominance chains are, in fact, direct; of length 1. The dashed arrow now reflect dominance between the elements in the sets. In particular, there is no dominance between the elements in $P_1(x)$ and $P_1(y)$ since when attaching $(h, x) \in P_1(x)$ the outcome y, the phase changes to P_2 and hence $(h, x, y) \in P_2(y)$. Similarly, there is no dominance between the elements in $P_2(z)$ and $P_1(x)$ since when attaching $(h, z) \in P_2(z)$ the outcome x, the phase does not change. Thus the construction is internally stable. External stability is easy to see: any element in $P_1(z)$ is dominated by an element in $P_1(x)$, etc..

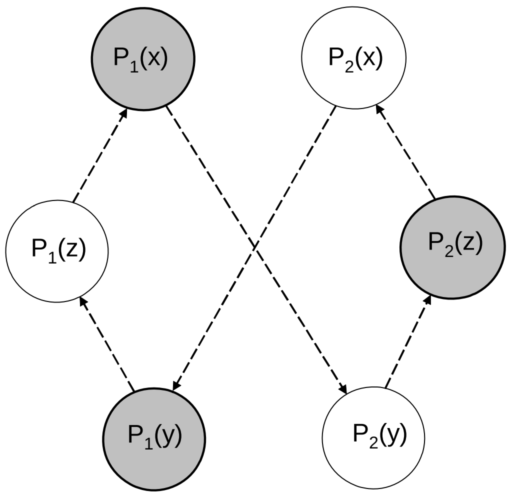


Figure 1.

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2 Dynamic stability

We assume that the set of social alternatives X is a *compact metric space*. The voting situation is described by a relation R. We denote the asymmetric part of R by P.

Denote the *lower contour set* and the *strict lower contour set* at x, respectively, by

$$L(x) = \{y \in X : xRy\} \text{ and } SL(x) = \{y \in X : xPy\}. \tag{1}$$

Also denote by $L^{-1}(x) = \{y \in X : x \in L(y)\}$ the corresponding *upper contour set* of x and $I(x) = L(x) \cap L^{-1}(x)$ there *indifference set* of x. Note that $I(x) = L(x) \cap L^{-1}(x)$ and $L^{-1}(x) = X \setminus SL(x)$. The correspondence L has a *closed graph* if $L(x)$ and $L^{-1}(x)$ (and hence $I(x)$) are closed for all $x \in X$.[5]

We make the following assumptions:

A1 R is complete.

A2 L has a closed graph.

A1 is met if the binary relation is governed by majority comparison. A2 is met e.g. if X is a finite set or if R is derived from the majority comparison in the spatial voting context.

We say that x *directly dominates* y if xPy and that x *weakly directly dominates* y if xRy. The motivation for the dominance structure is players voting behavior: y is preferred to x by the voters in a decisive coalition, and hence there is a tendency to move from implementing x to implementing y. However, farsighted voters should anticipate further deviations along the deviation path, and care only of the final outcome (Harsaynyi, 1970). To capture farsightedness, we extend the notion of dominance as follows: $(x_0, ..., x_K) \in X^K$ is a *weak* (indirect) *dominance chain* if $x_K R x_k$, for all $k = 0, ..., K-1$.

We have in mind the following dynamic procedure: there is a distinguished initial outcome x^*, serving as the initial status quo. Charatcerized by the relation R, individuals or coalitions - actors in general - participating the dynamic decision process, may challenge the current status quo by proposing a *different* alternative. That is, for any $k = 0, 1, ...$, after the sequence $x^* = x_0, ..., x_k$ of status quos, if $x_{k+1} = x_k$, then x_k becomes implemented. Call a finite set of successive non-implemented status quos as a *history*. Denote by $H = \{x^*\} \cup_{k=0}^{\infty} X^k$ the set of all finite histories.

Our quilibrium condition is given by the following:

Definition 1 (Dynamic stability) *A* dynamic stable set $V \subset H$ *is defined by:*

1. *(External farsighted stability) If $(h, x) \notin V$, then there is $(h, x, ..., y) \in V$ such that $(x, ..., y)$ is a weak dominance chain.*

[5] Since X is compact, the closed craph -condition of L implies that it is upper hemicontinuous as a correspondence.

2. *(Internal farsighted stability) If* $(h, x) \in V$*, then there is no* $(h, x, ..., y) \in V$ *such that* $(x, ..., y)$ *is a weak dominance chain.*

Dynamic stable set characterizes equilibrium behavior in a sense that a weak dominance chain is interpreted as the equilibrium law of motion. In no equilibrium, if an outcome is to be implemented, threre should be no equilibrium move away from this outcome. Furthermore, from any nonequilibrium there should be a lw of motion towards an equilibrium outcome. Thus the idea of weak dominance path reflecting a law of motion is free from inner contradictions.

We seek to give a simple characterization of the welfarist properties of the dynamic stable set, i.e. we concentrate on the outcomes that are implementable via a dynamic stable set. The final outcome x in a history (h, x) in a dynamic stable set V is interpreted to become implemented once the history is reached as no decisive coalition wants to challenge it given that the play eventually converges to back to V. Set Y of outcomes is said to be *implementable* with the dynamic stable set V if

$$Y = \{x : (h, x) \in V, h \in H\}.$$

The set of implementable outcomes is our main object of our study. The initial status quo may affect the eventual outcome that will become implemented in the set of implementable outcomes but not the set itself.

Robustness of the solution As discussed by Xue (1998), the underying assumptions behind farsighted dominance may be too ambitious. The problem is that a farsighted dominance chain need not be incentive compatible. Farsighted dominance requires that the final element (weakly) dominates the middle elements but not that the final element should be "optimal" among all the possible continuation paths from a middle element perspective. Hence there is no guarantee that all the players moving in the middle of a chain will adhere to the plan. Because of the implicit optimism embedded into the notion of indirect dominance, also the dynamic stable set could be in danger. For if an active coalition cannot be sure that the projected outcome is eventually reached, it may no longer be in the coalition members' interest to participate the coalition. But then an element outside the stable set need no longer be unstable which invalidates the indirect dominance argument. Furthermore, if an element outside the stable set fails to be unstable, then there is no guarantee that some coalition will not challenge an outcome inside the stable set by demanding this outcome. Hence *also* internal stability may become invalidated.

However, in under our assumptions this scepticism is not warranted since, by A1, weak indirect dominance implies weak *direct* dominance. By the definition of direct dominance, the first deviating coalition can reach the final element in the deviation path in a single step. Thus if a coalition wants to initiate a weak dominance chain in the hope of eventually implementing the final element, he can as well move to the desired outcome directly. Thus there need not be a question about the credibility of the interim deviations. Moreover, since the dynamic stable set is defined with respect to indirect dominance, it is still true that any outcome outside the stable set is directly dominated by an outcome in the set. Thus a deviation from the stable set leads back, in one step, to the stable set. By the definition of indirect dominance, the latter outcome is *not* preferred by all the members of the originally deviating coalition and hence the Harsanyi-critique is avoided. Both internal and external stability should hold without reservations.

In fact, any dynamic stable set is also a dynamic *direct* stable set $V \subset H$,defined by the following conditions:

1. (External direct stability) If $(h, x) \notin V$, then there is $(h, x, y) \in V$ such that xRy.

2. (Internal direct stability) If $(h, x) \in V$, then there is no $(h, x, y) \in V$ such that xRy.

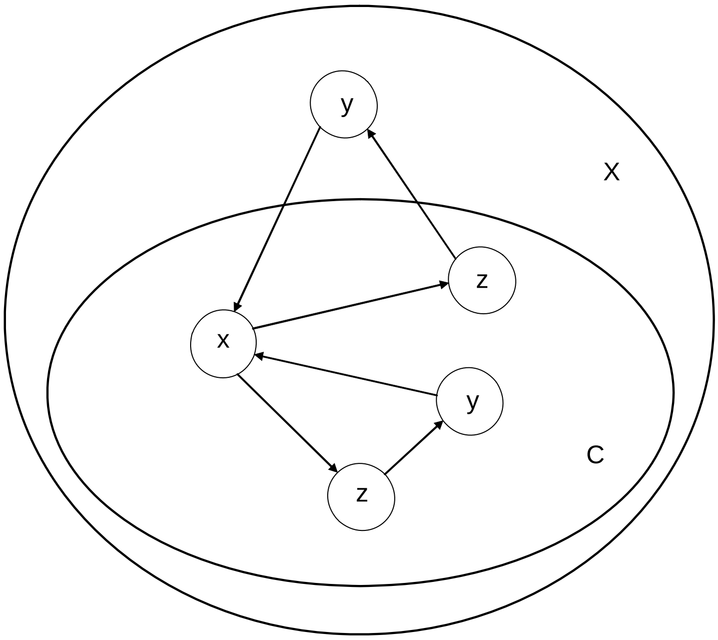
However, the converse is not true. But it is this double-stability feature that makes the farsighed version of the dynamic stable set credible in the current setting. Of course, this double-stability feature requires completeness of the dominance relation R.

We now give a straightforward characterization of the outcomes that are implementable with a dynamic stable set.

2.1 Characterization

Definition 2 (Consistent choice set) *A nonempty set $C \subseteq X$ is a* consistent choice set *if, for any $x \in C$, $y \notin L(x)$ implies that there is $z \in C$ such that $z \in L(x) \setminus SL(y)$.*

That is, if an element in a consistent choice set C is dominated by an (any) alternative, then this alternative is itself dominated by an alternative in C that does *not* dominate the original element in C. A consistent choice set meeting Definition 2 is related to the consistent set defined by Chwe (1994) [discussion to be added].



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Now we establish that any dynamic stable set (would such exist) is outcome equivalent to a consistent choice set.

Lemma 3 *V is a dynamic stable set only if $\{x : (h, x) \in V,\ h \in H\}$ is a consistent choice set.*

Proof. Let V be a stable set. We show that $\{x' : (h', x') \in V,\ h' \in H\}$ meets Definition 2. Take $x \in \{x' : (h', x') \in V, h' \in H\}$. Identify $h \in H$ such that $(h, x) \in V$. Suppose that there is (h, x, y) such that yPx, i.e. $y \notin L(x)$. Then (x, y) is a weak dominance chain. By internal stability, $(h, x, y) \notin V$. By external stability, there is $(h, x, y, ..., z) \in V$ such that $(y, ..., z)$ is a weak dominance chain. Thus $z \in \{x' : (h', x') \in V,\ h' \in H\}$ and zRy, or $z \notin SL(y)$. By internal stability, however, $(x, y, ..., z)$ cannot be a weak dominance chain implying - since $(y, ..., z)$ is a weak dominance chain - that xPz, or $z \in L(x)$, as desired. ■

Now we show the converse, that for any consistent choice set there exists a dynamic stable set that is outcome equivalent to the consistent choice set. For this purpose, we construct a stable set.

Fix a consistent choice set C. Let

$$Q = \{q^x : x \in C\}, \tag{2}$$

be a partition of H, constructed recursively as follows: Choose the initial outcome $x^* = x_0$. For any $(x_0, ..., x_k) = h \in H$, if $h \in q^x$, then

$$(h, y) \in \begin{cases} q^y, & \text{if } y \in SL(x) \cap C \text{ or } y = x, \\ q^x, & \text{if } y \notin SL(x) \cap C \text{ and } y = x. \end{cases} \tag{3}$$

Proceeding recursively this way for all $y \in C$ and all $h \in H$, the set of histories H is partitioned by Q.

Note that the transition rule (3) can be described directly via a transition function τ such that

$$\tau(q^x, y) = \begin{cases} q^y, & \text{if } y \in SL(x) \cap C \text{ or } y = x, \\ q^x, & \text{if } y \notin SL(x) \cap C \text{ and } y \neq x. \end{cases} \tag{4}$$

That is, whenever the play is in state q and the status quo is y, the new state is $\tau(q, y)$. We will use this specification in the proof below.

We let the agents to implement a the status quo y in state q^x if y is contained in $L(x) \cap C$. The procedure V^C corresponding to this idea is defined by

$$V^C = \{(q^x, y) : y \in SL(x) \cap C \text{ or } y = x\}. \tag{5}$$

By construction,

$$\begin{aligned} \{y : (q^x, y) \in V^C\} &= \{y : y \in SL(x) \cap C \text{ or } y = x, \text{ for some } x \in C\} \\ &= \{y : y = x, \text{ for some } x \in C\} \\ &= C. \end{aligned} \tag{6}$$

Thus, elements in the consistent choice set C are implementable with the procedure V^C. We next show that V^C is a stable set.[6]

Lemma 4 *V^C is a dynamic stable set.*

Proof. *External stability*: Take any $(q^x, y) \notin V^C$. There are two cases.

If $y \notin L(x) \cap C$, we show that there is z such that $z \in C \cap L(x) \setminus SL(y)$. (i) If $y \notin L(x)$, then $C \cap L(x) \setminus SL(y)$ is nonempty by the definition of C. (ii) If $y \in L(x)$, then $x \in C \cap L(x) \setminus SL(y)$. Thus there is always $z \in C$ such that, by the construction of τ, $(\tau(q^x, y), z) = (q^x, z)$ (directly) weakly dominates (q^x, y), and such that, by the construction of V^C, $(q^x, z) \in V^C$.

If $y \in I(x) \cap C$ and $y \neq x$, then $(\tau(q^x, y), x) = (q^x, x)$ (directly) weakly dominates (q^x, y) and $(q^x, x) \in V^C$.

Internal stability: Take any $(q^x, y) \in V^C$. For any z, by the construction of τ and V^C, if $(\tau(q^x, y), z) = (q^y, z) \in V^C$, then $z \in SL(y) \cap C$ or $z = y$. In the former case, (q^y, z) does not (directy) dominate (q^x, y). In the latter case, y becomes implemented. ■

By Lemma 3, a set Y of alternatives is implementable via a dynamic stable set only if Y is a consistent choice set. Conversely, by (6) and Lemma 4, outcomes of any consistent choice can be implemented via a stable set. We compound these observations in the following characterization.

[6] With the notational simplification that $(h, b) \in V^X$ if and only if $(q_a, b) \in V^X$ and $h \in q^a$.

Theorem 5 *Set Y of alternatives is implementable via a dynamic stable set if and only if Y is a consistent choice set.*

This result does not, however, tell anything about the existence of a consistent choice set nor how it can be identified. The next section provides an algorithm for identifying the maximal consistent choice set. Hence it also guarantees the existence of a solution.

2.2 Uncovered set

Given $B \subseteq X$, we say that y (strongly) *covers x in B* if (i) $x, y \in B$, (ii) yPx, and (iii) $z \in B$ and zRy implies zPx. By complenetess of R we can state this more compactly: x covers y in B, $x, y \in B$, if $L(x) \cap B \subset SL(y) \cap B$.[7]

The covering relation in B is transitive. Denote the maximal elements of the covering relation by $UC(B)$, the *uncovered* set of B (cf. Fishburn, 1977; Miller, 1980). That is $UC(B)$ comprises of the set of alternatives that are not covered in B by any element in B.

By Bordes et al. (1992) it is known that the uncovered set $UC(X)$ is nonempty.

Lemma 6 *Let B be a closed subset of X. Then $UC(B)$ is nonempty and closed.*

Proof. Since X is compact, B is compact. Since the covering relation is transitive, $UC(B)$ is nonempty if the covering relation attains its maximum in B. Since B is closed, there is a $z \in B$ such that $z_k \to z$ for $\{z_k\}$ such that $L(z_k) \cap B \subset SL(z_{k+1}) \cap B$ and $z_{k+1} \notin L(z_k)$ for all k. Since z_0 is arbitrary element of B, it suffices to show that z covers z_0. By induction, $L(z_0) \cap B \subset \cap_{k=0}^{\infty} SL(z_k) \cap B$. Since L has closed graph, $L(z) \cap B = \cap_{k=0}^{\infty} L(z_k) \cap B$. Thus $L(z_0) \cap B \subset SL(z) \cap B$, as desired.

Suppose $UC(B)$ is not closed. Then there is $\{x_k\} \subset UC(B)$ and $x \notin UC(B)$ such that $x_k \to x$. Since x is covered in B there is $y \in B$ such that $L(x) \cap B \subset SL(y) \cap B$, or $L(x) \cap L^{-1}(y) \cap B = \emptyset$. Take any sequence $\{y_k\} \subset B$ such that $y_k \to y$. By construction, $L(x_k) \cap B \not\subset SL(y_k) \cap B$ for all k. That is, there is z_k such that $z_k \in L(x_k) \cap L^{-1}(y_k) \cap B$ for all k. Find a converging subsequence $\{z_{k(j)}\}_j$ and $z \in B$ such that $z_{k(j)} \to_j z$. Then also $x_{k(j)} \to_j x$ and $y_{k(j)} \to_j y$. Since L has a closed graph, $z \in L(x) \cap L^{-1}(y) \cap B$. But then y does not cover x in B, a contradiction. ■

The iterated version of the uncvered set, the *ultimate uncovered set* is defined recursively and is analysed in the finite case e.g. by Miller (1980), Dutta (1988) and Laslier (1998). Infinite case has not, to the best of my knowledge, been analysed before.

[7] Since T is asymmetric and complete, a covers b iff $L(b) \cap B \subseteq L(a) \cap B$ and $a \neq b$.

Define the iterated covering procedure: Set $UC^0 = X$, and let $UC^{k+1} = UC(UC^k)$, for all $k = 0, ...$ Since a closed subset of a compact metric space is itself a compact metric space, UC^{k+1} exists for all $k = 0, ...$, by Lemma 6. Let $Z := UC^{\infty}$ which is nonempty and closed as well. Z is the ultimate uncovered set

Lemma 7 *Z exists and is closed.*

By construction, no element in Z is covered in Z.

Lemma 8 *Let $y \in X \diagdown Z$. Then there is $z \in Z$ such that $z \notin L(y)$ and $L(y) \cap Z \subset SL(z) \cap Z$.*

Proof. Choose $y = z_0$ and, for all $j = 0, ...$, find k_j such that z_{j+1} covers z_j in UC^{k_j} and $z_j \in UC^{k_j} \setminus UC^{k_j+1}$.

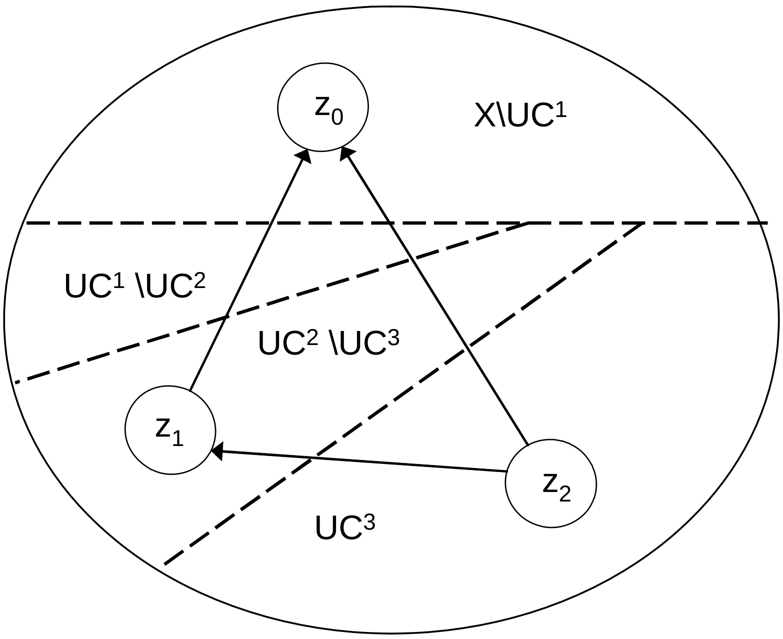

Figure 1.

Since $L(z_0) \cap UC^{k_0} \subset SL(z_1) \cap UC^{k_0}$, and since $UC^{k_1} \subseteq UC^{k_0}$, it follows that $L(z_0) \cap UC^{k_1} \subset SL(z_1) \cap UC^{k_1}$. As the same relation holds for z_1 and z_2, we have, by chaining the relations, $L(z_0) \cap UC^{k_2} \subset SL(z_2) \cap UC^{k_2}$ (depicted in Figure 1). By induction on $0, ..., j$, it follows that

$$L(z_0) \cap UC^{k_j} \subset SL(z_j) \cap UC^{k_j}. \tag{7}$$

Since X is compact there is z such that for a subsequence $\{z_k\}$ of $\{z_j\}$ we have $z_k \to z$. Since $z_k \in UC^k$ and UC^k is closed for all k, it follows that

$z \in \cap_{k=0}^{\infty} UC^k = Z$. By (7), $L(z_0) \cap Z \subset SL(z_k) \cap Z$, for all k. By the definition of interior, $SL(z_k) \cap Z \subset L(z_k) \cap Z$, for all k. Thus $L(z_0) \cap Z \subset L(z_k) \cap Z$, for all k. Finally, $L(z_0) \cap Z \subset L(z) \cap Z$ since L has closed graph. ■

The aim of this subsection is to prove the existence of a consistent choice set and, by Theorem **??**, that of a dynamic stable set. We now show that the ultimate uncovered set is a consistent choice set.

Theorem 9 *Z is a consistent choice set.*

Proof. Take $x \in Z$ and let $y \notin L(x)$. We find an element z in Z such that $z \in L(x) \setminus SL(y)$. If $y \in Z$, then such element exists by the definition of Z. Thus let $y \notin Z$.

By Lemma 8, there is $z \in Z$ such that $L(y) \cap Z \subset SL(z) \cap Z$. Since $z \notin L(y)$, we are done if $z \in L(x)$. Suppose, to the contrary, that $z \notin L(x)$. Since $x, z \in Z$, and $UC^{\infty} = Z$, it follows that $L(x) \cap Z \not\subset SL(z) \cap Z$. Thus, there is $w \in Z$ such that $w \in L(x) \setminus SL(z)$. Since $L(y) \subset SL(z)$, and $w \notin SL(z)$, we have that $w \notin L(y)$. Thus $w \in L(x) \setminus SL(y)$, as desired (see Figure 2).

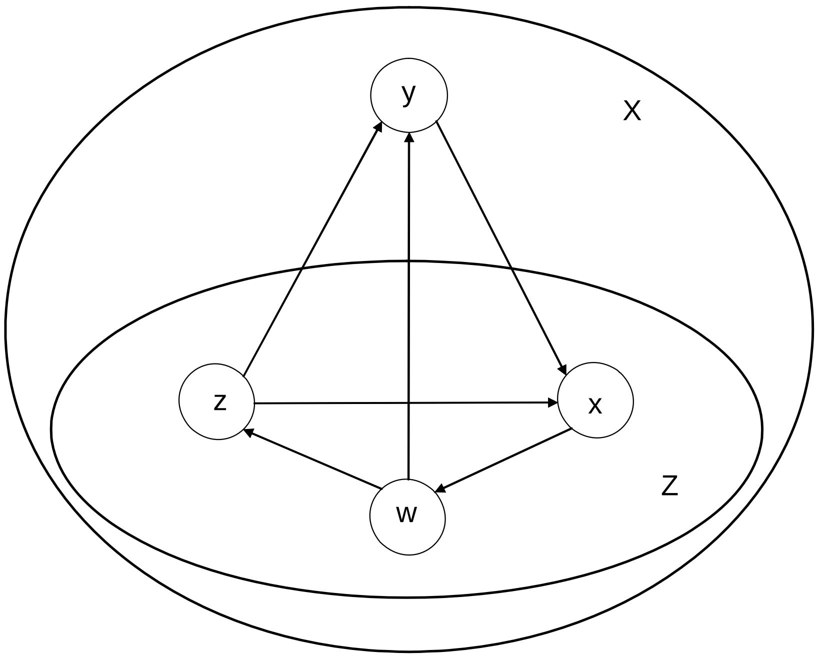

Figure 2.

■

The next result shows that Z is the (unique) maximal consistent choice set in the sense of set inclusion, given the asymmetric binary relation R.

Theorem 10 *Z is the maximal consistent choice set.*

Proof. Let C be a consistent choice set. We show that $C \subseteq Z$. Recall Definition 2: if $x \in C$ and $y \in X \setminus L(x)$, then $C \cap L(x) \setminus SL(y)$ is nonempty. Equivalently, for all $B \subseteq X$ such that $C \subseteq B$,

$$L(x) \cap B \not\subset SL(y) \cap B, \quad \text{for all } x \in C, \text{ for all } y \in B. \tag{8}$$

Since $C \subseteq X$, (8) implies that $L(x) \not\subset SL(y)$, for all $y \in C$. Thus, by the definition of covering in X, $C \subseteq UC(X)$. Again, by (8), $L(y) \cap UC(X) \not\subset SL(x) \cap UC(X)$ for all $x \in C$, for all $y \in UC(X)$. Thus, by the definition of covering in $UC(X)$, $C \subseteq UC(UC(X)) = UC^2$. By induction, $C \subseteq UC^{\infty} = Z$. ■

By Lemma 4 and Theorem 9, V^Z is a stable set. Moreover, by Theorem 9, the outcomes induced by V^Z are the maximal set of outcomes induced by any dynamic stable set.

Corollary 11 *V^Z is a stable set. Moreover, Z is the unique maximal set of outcomes that can be implemented in any dynamic stable set.*

Thus it is without loss of generality to focus on Z if one is interested interested on the welfare consequences of dynamically stable coalitional bargaining with farsighted agents.

3 Conclusion

The von Neumann-Morgenstern stable set solution can, in principle, be applied to any abstract decision theoretic situation with a proper dominance structure. The dominance structure can stem from any underlying motivation but it usually reflects a dynamic tendency. If one alternative is dominated by another since a decisive coalition wants to do that, then there is tendency to move towards the dominant outcome. This tendency is, of course, conditional on what to expect after the move has been accomplished. The stable set communicates what outcomes can be expected to be stable in the light of this dynamics.

One of the key problems with the stable set solution has been its lack of general existence properties. Indeed it has been argued that the stable set is too sensitive to be really useful (see Shubik, 1997). Another problem is that the solution - when it exists - is not easily computable. For these reasons and others, many less demanding solutions have been developed applied to study coalition formation (see e.g. Dutta, 1988; and Moulin, 1986).

Focusing on tournament structure, this paper argues that the existence problem stems from an implicit stationarity assumption. The behavioral tendencies that govern the underlying dynamics are allowed to depend only on the status quo outcome and not on the path that led to the current status quo. This paper develops a dynamic stable set that allows dominance, i.e., the strategies that govern the dynamics, to be history dependent. We show that a dynamic stable set always exists. Moreover, the ultimate uncovered set produces the (unique) maximal set of outcomes that are implementable via any dynamic stable set.

Dynamic stable set need not work outside the tournament structure. Vartiainen (2008) constructs a new solution that is based on one-deviation principle that solves some of the problems.

References

[1] Banks, J (1985), Sophisticated voting outcomes and agenda control, Social Choice and Welfare 1, 295-306

[2] Barberà, S. and A. Gerber (2007), A note on the impossibility of a satisfactory concept of stability for coalition formation games, Economics Letters 95, 85-90.

[3] Chwe, M. (1994), Farsighted stability, Journal of Economic Theory 63, 299-325.

[4] Coughlan, P. and M. LeBreton (1999), A social choice function implementable via backwards induction with values in the ultimate uncovered set, Review of Economic Design 4, 153-60.

[5] Downs, A. (1957) An Economic Theory of Democracy, New York: Harper and Row.

[6] Dutta. B., Jackson, M. and M. LeBreton (2002), Voting by successive elimination and strategic candidacy, Journal of Economic Theory 103, 190-218.

[7] Dutta. B (1988), Covering sets and a new condorcet correpondence, Journal of Economic Theory 44, 63-80.

[8] Fishburn, P. (1977), Condorcet social choice function, SIAM Journal of Applied Math 33, 295-306.

[9] Greenberg, J. (1990), Theory of social situations, Cambridge UP

[10] Harsanyi, J. (1974), An equilibrium point interpretation of stable sets and a proposed alternative definition, Management Science 20, 1427-95.

[11] Laslier, J.-F. (1997), Tournament Solutions and Majority Voting, Heidelberg, New-York: Springer-Verlag.

[12] McKelvey, R. (1986), Covering, dominance, and institution-free properties of social choice.American Journal of Political Science, 283-314

[13] Miller, N. (1980), A new solution set to for tournaments and majority voting, American Journal of Political Science 24, 68-96.

[14] Moulin, H. (1986), Choosing from a tournament, Social Choice and Welfare 3, 271-91.

[15] Penn, M. (2006), Alternate definitions of the uncovered set, and their implications, Social Choice and Welfare

[16] Patty, J and M. Penn (2007), Needing and covering, manuscript, Harvard

[17] Shepsle, K. and B. Weingast (1984), Uncovered sets and sophisticated outcomes with implications for agenda institutions, American Journal of Political Science 28, 49-74.

[18] Schofield, N. (1983). Generic instability of majority rule. Review of Economic Studies 50, 695-705

[19] Shubik, M. (1997), Game Theory, Complexity and Simplicity Part I: A Tutorial, Complexity 3, 39-46.

[20] Vartiainen, H. (2008), One-deviation solution in perfect information games, manuscript, Yrjö Jahnsson Foundation .

[21] Xue, L (1998), Coalitional stability under perfect foresight, Economic Theory 11, 603-27.

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