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ABSTRACT

In this paper, we study the optimal stopping problem of Dupuis and Wang analyzed in [7]. In this problem, the underlying follows a linear diffusion but the decision maker is not allowed to stop at any time she desires but rather on the jump times of an independent Poisson process. In [7], the authors solve this problem in the case where the underlying is a geometric Brownian motion and the payoff function is of American call option type. In the current study, we will this problem under weak assumptions on both the underlying and the payoff. We also demonstrate that the results of [7] are recovered from ours.

JEL Classification: C61

Keywords: Optimal stopping, linear diffusion, free boundary problem, Poisson process

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1. Introduction and the main result

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be a complete filtered probability space satisfying the usual conditions (see [2], p. 2). We assume that the state process X is a regular linear diffusion evolving on $\mathbf{R}_+$. Moreover, we assume that X does not die inside the state space and that the basic characteristics of X, namely the scale function S, the speed measure m and the killing measure k (see [2], pp. 13–14), are absolutely continuous with respect to the Lebesgue measure, have smooth derivatives and that the scale function S is twice continuously differentiable. Under these assumptions, we know that the infinitesimal generator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to C_b(\mathbf{R}_+)$ of X can be expressed as

$$\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx} - c(x),$$

where the functions σ, μ and c (the *infinitesimal parameters* of X) are related to m, k and S via the formulæ $m(x) = \frac{2}{\sigma^2(x)}e^{B(x)}$, $S'(x) = e^{-B(x)}$, and $k(x) = \frac{2c(x)}{\sigma^2(x)}e^{B(x)}$ for all $x \in \mathbf{R}_+$, where $B(x) := \int^x \frac{2\mu(y)}{\sigma^2(y)}dy$ (see [2], pp. 17). The assumption that the state space is $\mathbf{R}_+$ is done for reasons of notational convenience. In fact, we could assume that the state space is any interval $\mathcal{I}$ in $\mathbf{R}$ and all our subsequent analysis would hold true. Furthermore, we denote, respectively, as ψ and φ the increasing and the fundamental solution of the ordinary second-order linear differential equation $\mathcal{A}u = ru$, where $r > 0$, defined on the domain of the characteristic operator of X (for the characterization and fundamental properties of the minimal r-excessive functions, ψ and φ, see [2], pp. 18–20). We assume that the filtration $\mathcal{F}$ is rich enough to carry a Poisson process $N = (N_t, \mathcal{F}_t)$ with intensity λ – we call the process N the *signal process*, the intensity λ the *information rate*, and assume that X and N are *independent.* Later we will also need the increasing and decreasing solutions of the differential equation $\mathcal{A}u = (r+\lambda)u$, these solutions will be denoted as ψ_λ and φ_λ.

For $r > 0$, we denote as L_1^r the class of real valued measurable functions f on $\mathbf{R}_+$ satisfying the condition

$$\mathbf{E}_x\left[\int_0^\zeta e^{-rt}\,|f(X_t)|\right] < \infty, \tag{1.1}$$

where ζ denotes the lifetime of the state process X. For a function $f \in L_1^r$, the *resolvent* $R_r f : \mathbf{R}_+ \to \mathbf{R}$ is defined as

$$(R_r f)(x) = \mathbf{E}_x\left[\int_0^\zeta e^{-rs} f(X_s)ds\right] \tag{1.2}$$

for all $x \in \mathbf{R}_+$. The resolvent R_r and the increasing and decreasing solutions ψ and φ are closely connected in a computationally very useful way. Indeed, we from the literature that for any $f \in L_1^r$ the resolvent $R_r f$ can be expressed as

$$(R_r f)(x) = B^{-1}\varphi(x)\int_0^x \psi(y)f(y)m'(y)dy + B^{-1}\psi(x)\int_x^\infty \varphi(y)f(y)m'(y)dy \tag{1.3}$$

for all $x \in \mathbf{R}_+$, where $B = \frac{\psi'(x)}{S'(x)}\varphi(x) - \frac{\varphi'(x)}{S'(x)}\psi(x)$ denotes the constant Wronskian determinant (see [2], pp. 19).

Having the underlying dynamic structure set up, we will now formulate, following [7], the optimal stopping problems. In comparison to the ordinary continuous time case (see, e.g., [1], [5], [8] and [13]), the key difference is that the decision is not allowed to stop at any time she chooses but rather on the jump times of the independent signal process N. The process N jumps at times $T_1 < T_2 < \cdots < T_n < \dots$, where the intervals $\{T_1, T_2 - T_1, T_3 - T_2, \dots\}$ are exponentially IID. We remark that by convention $T_0 = 0$ and $T_\infty = \infty$.

In the first optimal stopping problem, the decision maker cannot stop at the initial time $t = 0$. This means that the time of the first jump T is the first potentially reasonable moment for her to exercise. In this setting, the class of admissible stopping times reads as

$$\mathcal{T} = \{\tau : \text{ for all } \omega \in \Omega,\ \tau(\omega) = T_n(\omega) \text{ for some } n \in 1, 2, \dots, \infty\}. \tag{1.4}$$

Let $r > 0$ be the discount rate and $g : \mathbf{R}_+ \to \mathbf{R}$ the exercise payoff function. At this stage, we assume that $g \in L_1^r$. The first optimal stopping problem is now to maximize the expected present value of the exercise payoff under $\{\mathcal{F}_\tau\}_{\tau \in \mathcal{T}}$, i.e. to determine the optimal value function

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbf{E}_x \left[e^{-r\tau} g(X_\tau) \right] \tag{1.5}$$

and to characterize the optimal stopping time τ^* constituting this value.

The second optimal stopping problem is otherwise the same as the first but now the decision maker can stop $t = 0$. Since it can very well be reasonable for her to stop immediately, the class of admissible stopping times in this alternate setting reads as

$$\mathcal{T}_0 = \{\tau \ : \ \text{ for all } \omega \in \Omega,\ \tau(\omega) = T_n(\omega) \text{ for some } n \in 0, 1, 2, \dots, \infty\}. \tag{1.6}$$

The corresponding optimal stopping problem reads as

$$V_0(x) = \sup_{\tau \in \mathcal{T}_0} \mathbf{E}_x \left[e^{-r\tau} g(X_\tau) \right] \tag{1.7}$$

and the optimal stopping time is denoted as τ_0^*.

These optimal stopping problems were first proposed by Dupuis and Wang in [7]. In this paper they solve the special case where the underlying follows a geometric Brownian motion and the payoff function is of American call option type $x \mapsto (x - K)^+$, with $K > 0$. In this paper we prove a generalization of their result formulated in the next theorem, which is, at least to the authors best knowledge, a new result.

Theorem 1.1. *Assume, that there is a unique state $\hat{x}$ which maximizes the function $x \mapsto \frac{g(x)}{\psi(x)}$ and that the function $x \mapsto \frac{g(x)}{\psi(x)}$ is nondecreasing on $(0, \hat{x})$ and* *nonincreasing on* $(\hat{x},\infty)$*. Then the threshold* $x^* < \hat{x}$ *characterized uniquely by the condition*

$$\psi(x^*)\int_{x^*}^{\infty}\varphi_\lambda(y)g(y)m'(y)dy = g(x^*)\int_{x^*}^{\infty}\varphi_\lambda(y)\psi(y)m'(y)dy$$

constitutes the optimal stopping rule for the optimal stopping problems (1.5) *and* (1.7)*. Moreover, the optimal value functions* V *and* V_0 *can be written as*

$$V(x) = \begin{cases} \lambda(R_{r+\lambda}g)(x) + \frac{g(x^*)-\lambda(R_{r+\lambda}g)(x^*)}{\varphi_\lambda(x^*)}\varphi_\lambda(x), & x \geq x^* \\ \frac{g(x^*)}{\psi(x^*)}\psi(x), & x < x^* \end{cases} \tag{1.8}$$

and

$$V_0(x) = \begin{cases} g(x), & x \geq x^* \\ \frac{g(x^*)}{\psi(x^*)}\psi(x), & x < x^*. \end{cases} \tag{1.9}$$

We remark that the assumptions of 1.1 are essentially the same as in [1], where the problem is studied in the ordinary continuous time setting. In this sense, it is interesting to note that the restriction of the admissible stopping times from continuum to random times with exponential arrivals does not result into any additional restrictions on the underlying and the payoff – the "degree of solvability" remains the same. We also remark that the class of problems considered in this study is related to the job search problem, see, e.g., [15], [4] and [9]. In this problem, the person is facing a sequence of job offers with random arrivals and the goal of this person is to maximize the expected present value of the return obtained by accepting the job offer.

The original interpretation of the optimal stopping problems in [7] is that the underlying is observable at all times but the decision maker can act only at jump times of the signal process. We remark that this interpretation can be turned the other way around such that that decision maker can act at all times but the state of the underlying is observed only at the jump times of N. In this setting, the observed sample paths are actually pure jump paths with jumps at times T_i and remaining constant in between – something reminiscent of a semi-Markov process. Consequently, this alternate interpretation *could* have some implications to the optimal stopping of semi-Markov processes, see, e.g., [3], [4] and [16]. But this something that is left for future research.

2. The proof of the main result

2.1. **Some preliminary analysis.** The assumptions of Theorem 1.1 restraining the choice of the payoff function g and the underlying X are relatively weak and easy to verify, given that we know the increasing fundamental solution ψ. In the ordinary case of continuous time stopping, we know that the ratio function $x \mapsto \frac{g(x)}{\psi(x)}$ and its monotonicity properties play a key role (see, e.g., [1]). In the current setting, it not the ratio $x \mapsto \frac{g(x)}{\psi(x)}$ but something at least formally quite reminiscent to it that characterizes the optimal stopping rules. To make a precise statement, define the functions $I : \mathbf{R}_+ \to \mathbf{R}$ and $J : \mathbf{R}_+ \to \mathbf{R}$ as

$$\begin{aligned} I(x) &= \int_x^\infty \varphi_\lambda(y) g(y) m'(y) dy, \\ J(x) &= \int_x^\infty \varphi_\lambda(y) \psi(y) m'(y) dy \end{aligned} \tag{2.1}$$

for all $x \in \mathbf{R}_+$. The ratio function $x \mapsto \frac{I(x)}{J(x)}$ will play the key role when analyzing the problems (1.5) and (1.7). The next lemma provides us with the required monotonicity properties of this function.

Lemma 2.1. *Let the assumptions of Theorem 1.1 hold. Then there is a unique state $x^* < \hat{x}$ that maximizes the function $x \mapsto \frac{I(x)}{J(x)}$. Moreover, the function $x \mapsto \frac{I(x)}{J(x)}$ is nondecreasing on $(0, x^*)$ and nonincreasing on $(0, x^*)$.*

Proof. We will begin the proof by deriving limiting properties for the function $x \mapsto \frac{I(x)}{J(x)}$. Since $\lim_{x\to\infty} I(x) = \lim_{x\to\infty} J(x) = 0$, L'Hospitals rule implies that $\lim_{x\to\infty} \frac{I(x)}{J(x)} = \lim_{x\to\infty} \frac{g(x)}{\psi(x)} = 0$. On the other hand, since $\lim_{x\to 0+} J(x) = \infty$, we prove completely analogously that $\lim_{x\to\infty} \frac{I(x)}{J(x)} = 0$. Now, straightforward differentiation yields the condition

$$\frac{d}{dx}\left(\frac{I(x)}{J(x)}\right) \gtreqqless 0 \text{ if and only if } \psi(x)I(x) \gtreqqless g(x)J(x).$$

Assume first that $x \geq \hat{x}$. Since the function $x \mapsto \frac{g(x)}{\psi(x)}$ is nonincreasing on $(\hat{x}, \infty)$, we find that

$$\begin{aligned} \psi(x)I(x) - g(x)J(x) &= \psi(x) \int_x^\infty \varphi_\lambda(y) \frac{g(y)}{\psi(y)} \psi(y) m'(y) dy - g(x)J(x) \\ &< \left(\psi(x) \frac{g(x)}{\psi(x)} - g(x) \right) J(x) = 0. \end{aligned}$$

We conclude that the function $x \mapsto \frac{I(x)}{J(x)}$ is nonincreasing on $(\hat{x}, \infty)$. This observation coupled with the limiting properties of $x \mapsto \frac{I(x)}{J(x)}$ imply that the function $x \mapsto \frac{I(x)}{J(x)}$ must have at least one interior maximum $\hat{x}$. Finally, since $\frac{g(x^*)}{\psi(x^*)} = \frac{I(x^*)}{J(x^*)}$ and the function $x \mapsto \frac{g(x)}{\psi(x)}$ is nondecreasing on $(0, \hat{x})$, we conclude that the maximum x^* must be unique. $\square$

In Lemma 2.1 we proved that the function $x \mapsto \frac{I(x)}{J(x)}$ has a unique global maximum x^*. We remark that x^* is the unique state satisfying the condition

$$\psi(x^*)I(x^*) = g(x^*)J(x^*); \tag{2.2}$$

this relation will be important.

We conclude the subsection by making a remark on the verification phase. Indeed, the continuous time formulations (1.5) and (1.7) are not that handy from the verification point of view. In order to remedy this, define the filtration $\mathcal{G}$ as $\mathcal{G}_n := \mathcal{F}_{T_n}$ for all $n \geq 0$ where T_i is the ith jump time of the signal process N, and, the $\mathcal{G}$-adapted process Z as $Z_n := (T_n, X_{T_n})$. Moreover, define the sets $\mathcal{N}$ and $\mathcal{N}_0$ as

$$\mathcal{N} = \{N \geq 1 : N \text{ is a } \mathcal{G}\text{-stopping time}\}$$
$$\mathcal{N}_0 = \{N \geq 0 : N \text{ is a } \mathcal{G}\text{-stopping time}\}.$$

Then Lemma 1 of [7] implies that the optimal stopping problems (1.5) and (1.7) can be formulated alternatively as

$$\begin{aligned} V(x) &= \sup_{N \in \mathcal{N}} \mathbf{E}\left[\tilde{g}(Z_N) | Z_0 = (0, x)\right] \\ V_0(x) &= \sup_{N \in \mathcal{N}_0} \mathbf{E}\left[\tilde{g}(Z_N) | Z_0 = (0, x)\right] \end{aligned} \tag{2.3}$$

for all $x \in \mathbf{R}$ where $\tilde{g}(Z_N) := e^{-rT_n} g(X_{T_n})$. Formulations (2.3) allow a straightforward usage of martingale techniques in the verification phase, as we will see later.

2.2. **The free boundary problem.** In this subsection we will start the analysis of the problems (1.5) and (1.7) by first tackling (1.5). This is done by proposing a suitable free boundary problem and solving it; for a recent exposition of free boundary methods in optimal stopping, see [11]. Following the lines of [7], we will proceed heuristically for the time being. The heuristics are the same as in [7], but for the sake of completeness we will present them here.

Typically in problems of the type (1.5), the optimal stopping rule is a threshold rule, be it one-sided or many-sided. We make an *ansatz* that the optimal stopping rule is a one-sided threshold rule of the form

"Stop at the first jump time T_i when the state variable X exceeds some predetermined threshold y^*".

Formally speaking, the optimal stopping time is then

$$\tau_{y^*} = \inf\{T_n, n \geq 1 : X_{T_n} \geq y^*\}.$$

We will denote as G the value function constituted by the stopping time τ_{y^*}. On the continuation region $(0, y^*)$ we would expect that G, our candidate for the optimal value, is r-harmonic. However, on the exercise region (y^*, ∞) the decision maker cannot stop until N jumps. In an infinitesimal time interval dt, the signal process N has probability λdt of making a jump. This means that in an infinitesimal time dt, the jump and, consequently, exercise with payoff $g(x)$, has probability λdt. On the other hand, the absence of jump forces the decsion maker to continue with probability $1 - \lambda dt$. Formally, this suggests with a heuristic use of Dynkin's theorem (see, e.g., [10]) that

$$\begin{aligned} G(x) &= g(x)\lambda dt + (1 - \lambda dt)\mathbf{E}_x[e^{-rdt}G(X_{dt})] \\ &= \lambda g(x)dt + (1 - \lambda dt)[G(x) + ((\mathcal{A} - r)G)(x)dt] \\ &= G(x) + (\mathcal{A} - r)G)(x)dt + \lambda(g(x) - G(x)) \end{aligned}$$

for all $x > y^*$ under the intuition $dt^2 = 0$. Finally, this yields the condition

$$(\mathcal{A}G)(x) - rG(x) + \lambda(g(x) - G(x)) = 0 \tag{2.4}$$

for all $x > y^*$. Moreover, we can expect that $g(x) < G(x)$ on $(0, y^*)$ and due to the possibility that N doesn't jump when $X \geq y^*$ that $G(x) < g(x)$ on (y^*, ∞). Typically in problems of the form (1.5) the value function is continuously differentiable over the optimal boundary. If we are willing except this, we readily verify using the r-harmonicity of G on the continuation region and condition (2.4) that $G(y^*) = g(y^*)$. Using these heuristics, we pose the following free boundary problem: F*ind a non-negative continuously differentiable function G and a unique state y^* satisfying the conditions*

$$\begin{cases} G(0+) \geq 0, \\ G(y^*) = g(y^*), \\ (\mathcal{A}G)(x) = rG(x) \text{ and } G(x) > g(x) & x < y^* \\ (\mathcal{A}G)(x) = rG(x) + \lambda(G(x) - g(x)) \text{ and } G(x) < g(x), & x > y^*. \end{cases} \tag{2.5}$$

The free boundary problem (2.5) will now be used as the device to an produce explicit candidate for the optimal stopping threshold x^* and the optimal value function V of the problem (1.5). This is done by deriving necessary conditions for the existence of a unique solution (G, y^*) of (2.5). Assume now that the unique solution (G, y^*) exists and that $x < y^*$. The condition $\mathcal{A}G = rG$ implies that there exists unique constants c_1 and c_2 such that $G(x) = c_1\psi(x) + c_2\varphi(x)$ for all $x < y^*$. Since we are looking for a bounded solution, c_2 must be equal to 0. Hence, $G(x) = c_1\psi(x)$ whenever $x < y^*$. Now, let $x \geq y^*$. The fourth condition in the free boundary problem (2.5) can be written as

$$(\mathcal{A}G)(x) - (r + \lambda)G(x) = -\lambda g(x). \tag{2.6}$$

for all $x > y^*$. A particular solution to the equation (2.6) is the resolvent $\lambda(R_{r+\lambda}g)(x)$ and, consequently, the general solution can be written as

$$G(x) = \lambda(R_{r+\lambda}g)(x) + d_1\psi_\lambda(x) + d_2\varphi_\lambda(x),$$

where ψ_λ and φ_λ the increasing and decreasing solutions of the homogenous equation $(\mathcal{A}G)(x) - (r+\lambda)G(x) = 0$. Our standing assumption implies that the constant d_1 must be zero. Hence $G(x) = \lambda(R_{r+\lambda}g)(x) + d_2\varphi_\lambda(x)$ for all $x > y^*$. Since G is continuous, the equalities $g(y^*) = c_1\psi(y^*) = \lambda(R_{r+\lambda}g)(y^*) + d_2\varphi_\lambda(y^*)$ must hold. This implies that

$$c_1 = \frac{g(y^*)}{\psi(y^*)}, \; d_2 = \frac{g(y^*) - \lambda(R_{r+\lambda}g)(y^*)}{\varphi_\lambda(y^*)}$$

and, consequently, that the value G can be expressed as

$$G(x) = \begin{cases} \lambda(R_{r+\lambda}g)(x) + \frac{g(y^*)-\lambda(R_{r+\lambda}g)(y^*)}{\varphi_\lambda(y^*)}\varphi_\lambda(x), & x \geq y^* \\ \frac{g(y^*)}{\psi(y^*)}\psi(x), & x < y^* \end{cases}$$

Since the function G is continuously differentiable over the boundary y^*, we observe that

$$g(y^*)\frac{\psi'(y^*)}{\psi(y^*)} - \lambda(R_{r+\lambda}g)'(y^*) - \frac{g(y^*) - \lambda(R_{r+\lambda}g)(y^*)}{\varphi_\lambda(y^*)}\varphi_\lambda'(y^*) = 0, \tag{2.7}$$

which can be rewritten as

$$g(y^*)\left(\frac{\psi'(y^*)}{\psi(y^*)} - \frac{\varphi_\lambda'(y^*)}{\varphi_\lambda(y^*)}\right) = \lambda(R_{r+\lambda}g)'(y^*) - \frac{\varphi_\lambda'(y^*)}{\varphi_\lambda(y^*)}\lambda(R_{r+\lambda}g)(y^*).$$

By invoking the representation (1.3) and straightforward differentiation, we find that the righthand-side can be expressed as

$$\lambda(R_{r+\lambda}g)'(y^*) - \frac{\varphi_\lambda'(y^*)}{\varphi_\lambda(y^*)}\lambda(R_{r+\lambda}g)(y^*) = \lambda\frac{S'(y^*)}{\varphi_\lambda(y^*)}\int_{y^*}^\infty \varphi_\lambda(y)g(y)m'(y)dy,$$

where S' is the scale density of the state process X. Consequently, the optimality condition (2.7) can be expressed as

$$\lambda\psi(y^*)\int_{y^*}^\infty \varphi_\lambda(y)g(y)m'(y)dy = g(y^*)\left(\frac{\psi'(y^*)}{S'(y^*)}\varphi_\lambda(y^*) - \frac{\varphi_\lambda'(y^*)}{S'(y^*)}\psi(y^*)\right).$$

Denote as $w(x) = \frac{\psi'(x)}{S'(x)}\varphi_\lambda(x) - \frac{\varphi_\lambda'(x)}{S'(x)}\psi(x)$. By applying the differential equations $\mathcal{A}\psi = r\psi$ and $\mathcal{A}\varphi_\lambda = (r+\lambda)\varphi_\lambda$ we find that $w'(x) = -\lambda\varphi_\lambda(x)\psi(x)m'(x)$. Now, Fundamental Theorem of Calculus implies that

$$w(y^*) = \lambda\int_{y^*}^\infty \varphi_\lambda(y)\psi(y)m'(y)dy.$$

and, consequently, that the optimality condition (2.7) can be expressed as

$$\psi(y^*)\int_{y^*}^\infty \varphi_\lambda(y)g(y)m'(y)dy = g(y^*)\int_{y^*}^\infty \varphi_\lambda(y)\psi(y)m'(y)dy. \tag{2.8}$$

But now, we established in Lemma 2.1 that the state x^* characterized as the unique maximum of the function $x \mapsto \frac{I(x)}{J(x)}$ is the unique state satisfying the condition (2.8) (see expression (2.2)). Having observed this, we have proved the following lemma.

Lemma 2.2. *Let the assumptions of Theorem 1.1 hold. Then the threshold x^* characterized uniquely by the condition* (2.8)*, and the function G defined as*

$$G(x) = \begin{cases} \lambda(R_{r+\lambda}g)(x) + \frac{g(x^*)-\lambda(R_{r+\lambda}g)(x^*)}{\varphi_\lambda(x^*)}\varphi_\lambda(x), & x \geq x^* \\ \frac{g(x^*)}{\psi(x^*)}\psi(x), & x < x^* \end{cases} \tag{2.9}$$

form the unique solution for the free boundary problem (2.5)*.*

Lemma 2.2 show that under our standing assumptions, the free boundary problem (2.5) has a unique solution. Following the heuristics of the beginning of this section, the threshold x^* constitutes a stopping rule, which should be optimal (in this heuristic sense) among all admissible stopping rules. Now the crucial question is that does the function G defined in (2.9) correspond to the exercise rule *"stop at the first jump time T_i when X exceeds x^*"* (i.e., to the stopping time τ_{x^*}) and, furthermore, are these characteristics G and τ_{x^*} optimal – i.e., is $V \equiv G$ and $\tau^* = \tau_{x^*}$?

2.3. **The verification phase.** The previous subsection was concerned with the optimal stopping problem (1.5). We justified heuristically the free boundary problem (2.5) which produced our candidates for the optimal stopping rule and the optimal value function for (1.5) – see Lemma 2.2. Our next task is to apply verification procedure to demonstrate that our candidates is not just any candidates but actually the optimal characteristics. In the process of doing this, we will first turn our attention to the other optimal stopping problem, namely the problem (1.7). The distinguishing feature in these problems is the initial information on X. In problem (1.7) the decision maker has this information and in problem (1.5) has not. We make an *ansatz* this added information on the initial state is so little that it does not change our candidate for the optimal exercise threshold x^*. However, it naturally does change candidate for the optimal value function, say G_0, so that G_0 must coincide with the exercise payoff g in the exercise region. Put formally, define the function $G_0 : \mathbf{R}_+ \to \mathbf{R}_+$ as

$$G_0(x) = \begin{cases} g(x), & x \geq x^* \\ \frac{g(x^*)}{\psi(x^*)}\psi(x), & x \leq x^*, \end{cases}$$

where x^* is the threshold uniquely determined by the condition (2.8). Now, G_0 is our candidate for the optimal value V_0 and τ_{x^*} is for the optimal exercise time. We will now proceed by proving this claim and then turn back to the first problem. The next technical lemma provides us with a useful connection of the functions G and G_0.

Lemma 2.3. *Let the assumptions of Theorem 1.1 hold. Then the function G can be expressed as $G(x) = \lambda(R_{r+\lambda}G_0)(x)$ for all $x \in \mathbf{R}_+$.*

Proof. The free boundary problem (2.5) implies that the function G satisfies the differential equation $(\mathcal{A}G)(x) - (r+\lambda)G(x) = \lambda G_0(x)$. Therefore we can express the function G as

$$G(x) = \lambda(R_{r+\lambda}G_0)(x) + c_1\psi_\lambda(x) + c_2\varphi_\lambda(x) \tag{2.10}$$

for all $x \in \mathbf{R}_+$. Assume that $x < x^*$. Then the expression (2.10) implies that

$$\frac{g(x^*)}{\psi(x^*)}\psi(x) - c_1\psi_\lambda(x) - \lambda(R_{r+\lambda}G_0)(x) = c_2\varphi_\lambda(x).$$

The left hand side of this expression stays bounded as $x \to 0+$. This implies that $c_2 = 0$. Assume now that $x > x^*$. Since G is a solution of the free boundary problem (2.5), we find that $c_1\psi_\lambda(x)+\lambda(R_{r+\lambda}G_0)(x)<g(x)$. Since this expression holds for all $x>x^*$, we conclude that $c_1=0$. $\square$

In Lemma 2.3 we prove that under our standing assumptions the solution of the free boundary (2.5) is a $(r+\lambda)$-potential. Using this lemma, we will turn to the verification phase for the problem (1.7). The first step is to show that G_0 constitutes a supermartingale.

Lemma 2.4. *Let the assumptions of Theorem 1.1 hold. Then the process*

$$S:=(e^{-rT_n}G_0(X_{T_n});\mathcal{G}_n)_{n\geq 0}$$

is a non-negative uniformly integrable supermartingale.

Proof. Lemma 2.3 implies that $G_0(x)\geq G(x)=\lambda(R_{r+\lambda}G_0)(x)=\mathbf{E}_x\left[e^{-rU}G_0(X_U)\right]$, where U is an independent exponential random time with rate λ. Thus the process S is a non-negative supermartingale. In order to prove uniform integrability, it is sufficient to show that S satisfies the conditions $\sup_n\mathbf{E}_x[S_n]<\infty$ and $\sup_n\mathbf{E}_x[S_n\mathbf{1}_A]\to 0$ as $\mathbf{P}(A)\to 0$ – if these conditions hold, the uniform integrability follows form [14], p. 190, Lemma 2.

Assume that $x\in\mathbf{R}_+$ and $n\geq 0$. Since the process $L:=\left(e^{-rT_n}\frac{\psi(X_{T_n})}{\psi(x)};\mathcal{G}_n\right)_{n\geq 0}$ is a martingale with $\mathbf{E}[L_n]=1$ for all $n\geq 0$, we define the measure $\mathbf{P}^*$ via the relation $\frac{d\mathbf{P}^*}{d\mathbf{P}}=L_n$. Denote as $\mathbf{E}^*$ the expectation under the measure $\mathbf{P}^*$ and let $A\in\mathcal{B}$. Since $\hat{x}$ is the global maximum of the function $x\mapsto\frac{g(x)}{\psi(x)}$, we find using the Radon-Nikodym theorem that

(2.11)
$$\begin{aligned}
\mathbf{E}_x[S_n\mathbf{1}_A]&=\psi(x)\mathbf{E}_x\left[\frac{V_0(X_{T_n})}{\psi(X_{T_n})}\mathbf{1}_AL_n\right]\\
&=\psi(x)\left\{\mathbf{E}_x^*\left[\frac{V_0(X_{T_n})}{\psi(X_{T_n})}\mathbf{1}_A\mathbf{1}_{\{X_{T_n}<x^*\}}\right]+\mathbf{E}_x^*\left[\frac{V_0(X_{T_n})}{\psi(X_{T_n})}\mathbf{1}_A\mathbf{1}_{\{X_{T_n}>x^*\}}\right]\right\}\\
&=\psi(x)\left\{\mathbf{E}_x^*\left[\frac{g(x^*)}{\psi(x^*)}\mathbf{1}_A\mathbf{1}_{\{X_{T_n}<x^*\}}\right]+\mathbf{E}_x^*\left[\frac{g(X_{T_n})}{\psi(X_{T_n})}\mathbf{1}_A\mathbf{1}_{\{X_{T_n}>x^*\}}\right]\right\}\\
&<\frac{g(\hat{x})}{\psi(\hat{x})}\psi(x)\mathbf{P}^*(A).
\end{aligned}$$

First, let $A=\mathbf{R}_+$ in the inequality (2.11). Since $\mathbf{E}_x[S_n]<\frac{g(\hat{x})}{\psi(\hat{x})}\psi(x)$ for all $n\geq 0$, we find that $\sup_n\mathbf{E}_x[S_n]\leq\frac{g(\hat{x})}{\psi(\hat{x})}\psi(x)<\infty$ for all $x\in\mathbf{R}_+$. On the other hand, since $\mathbf{P}$ and $\mathbf{P}^*$ are equivalent, we remark that $\mathbf{P}^*(A)\to 0$ whenever $\mathbf{P}(A)\to 0$. Thus, we conclude on the basis of (2.11) that $\mathbf{E}_x[S_n\mathbf{1}_A]\to 0$ for all $n\geq 0$ and, consequently, $\sup_n\mathbf{E}_x[S_n\mathbf{1}_A]\to 0$ that as $\mathbf{P}(A)\to 0$. $\square$

In Lemma 2.4 we show that under our standing assumptions, the process S is not only a non-negative supermartingale but also uniformly integrable. Uniform integrability will be needed in the proof of next lemma, where we use optional stopping with a stopping time which is not almost surely bounded.

Lemma 2.5. *Let the assumptions of Theorem 1.1 hold. Let* $\tau_0^* = T_{N_0^*}$ *where* $N_0^* = \inf\{n \geq 0 \ : \ X_{T_n} \geq x^*\}$. *Then*

$$G_0(x) = \mathbf{E}_x\left[e^{-r\tau_0^*}g(X_{\tau_0^*})\right] = V_0(x)$$

for all $x \in \mathbf{R}_+$.

Proof. Coupled with Lemma 2.4, optional sampling theorem now implies that $G_0(x) \geq \mathbf{E}_x\left[e^{-rT_N}G_0(T_N)\right] \geq \mathbf{E}_x\left[e^{-rT_N}g(T_N)\right]$ for all $\mathcal{G}$-stopping times N. Hence, $G_0(x) \geq V_0(x)$ for all $x \in \mathbf{R}_+$. We will proceed by proving that that the function G_0 can actually be attained by the admissible stopping rule *"stop at time* τ_0^**"*. To this end, we will first prove that the stopped process

$$Q = \left(e^{-rT_{N_0^* \wedge n}}G_0(X_{T_{N_0^*\wedge n}}); \mathcal{G}_n\right)_{n \geq 0}$$

is a martingale. The proof of the martingale property is completely analogous to the corresponding proof in [7] (see pp. 150), but we will review it here for the sake of complete presentation. For $n \geq 1$, we have

(2.12)

$$\begin{aligned}\mathbf{E}\left[Q_n|\mathcal{G}_{n-1}\right] &= \mathbf{E}\left[e^{-rT_n}V_0(X_{T_n})\mathbf{1}_{\{N_0^*\geq n\}}|\mathcal{G}_{n-1}\right] + \sum_{i=0}^{n-1}\mathbf{E}\left[e^{-rT_i}V_0(X_{T_i})\mathbf{1}_{\{N_0^*=i\}}|\mathcal{G}_{n-1}\right]\\ &= \mathbf{E}\left[e^{-rT_n}V_0(X_{T_n})\mathbf{1}_{\{N_0^*\geq n\}}|\mathcal{G}_{n-1}\right] + \sum_{i=0}^{n-1}e^{-rT_i}V_0(X_{T_i})\mathbf{1}_{\{N_0^*=i\}}.\end{aligned}$$

Denote as U an independent exponentially distributed random time with rate λ. Using the strong Markov property and Lemma 2.3, we find that the first term on the right hand side of (2.12) can be written as

$$\begin{aligned}\mathbf{E}\left[e^{-rT_n}V_0(X_{T_n})\mathbf{1}_{\{N_0^*\geq n\}}|\mathcal{G}_{n-1}\right] &= e^{-rT_{n-1}}\mathbf{E}_{X_{T_{n-1}}}\left[e^{-rU}V_0(X_U)\right]\mathbf{1}_{\{N_0^*\geq n\}}\\ &= e^{-rT_{n-1}}V(X_{T_{n-1}})\mathbf{1}_{\{N_0^*\geq n\}}.\end{aligned} \tag{2.13}$$

Finally, since $V_0(x) = V(x)$ when $x \leq x^*$, we conclude on the basis of the expressions (2.12) and (2.13) that

$$\mathbf{E}\left[Q_n|\mathcal{G}_{n-1}\right] = e^{-rT_{n-1}}V_0(X_{T_{n-1}})\mathbf{1}_{\{N_0^*\geq n\}} + \sum_{i=0}^{n-1}e^{-rT_i}V_0(X_{T_i})\mathbf{1}_{\{N_0^*=i\}} = Q_{n-1}.$$

Finally, since Q is also uniformly integrable, the result follows by optional sampling, i.e.,

$$G_0(x) = \mathbf{E}Q_{N_0^*} = \mathbf{E}_x\left[e^{-r\tau_0^*}G_0(X_{\tau_0^*})\right] = \mathbf{E}_x\left[e^{-r\tau_0^*}g(X_{\tau_0^*})\right] = V_0(x)$$

for all $x \in \mathbf{R}_+$. □

We proved in Lemma 2.5 that our candidates G_0 and τ_{x^*} for the optimal characteristics actually are the optimal characteristics of the problem (1.7). We will now turn back to the first optimal stopping problem and use the Lemmas 2.3-2.5 to prove that the candidates G and τ_{x^*} are the optimal characteristics of the problem (1.5).

Lemma 2.6. *Let the assumptions of Theorem 1.1 hold. Let* $\tau^* = T_{N^*}$ *where* $N^* = \inf\{n > 0 \ : \ X_{T_n} \geq x^*\}$. *Then*

$$G(x) = \mathbf{E}_x \left[e^{-r\tau^*} g(X_{\tau^*}) \right] = V(x)$$

for all $x \in \mathbf{R}_+$.

Proof. The representation (1.9) and the fact that x^* constitutes the optimal stopping rule for the optimal stopping rule (1.7) was proved already in Lemma 2.5. Since the process $(e^{-rT_n} G_0(X_{T_n}); \mathcal{G}_n)_{n \geq 0}$ is a non-negative supermartingale, we find that

$$\begin{aligned}\mathbf{E}_x \left[e^{-rT_n} g(X_{T_N}) \right] &\leq \mathbf{E}_x \left[e^{-rT_n} V_0(X_{T_N}) \right] \leq \mathbf{E}_x \left[e^{-rT_1} V_0(X_{T_1}) \right] = \lambda (R_{r+\lambda} G_0)(x) \\ &= G(x)\end{aligned}$$

for all $\mathcal{G}$-stopping times $N \geq 1$ and $x \in \mathbf{R}_+$. Taking supremum over all such N, we obtain the inequality $V(x) \leq G(x)$ for all $x \in \mathbf{R}_+$. It remains to show that the value G is the admissible stopping rule *"stop at time τ^*"*. By conditioning on the first jump time T_1 we find by using the strong Markov property and Lemma 2.5 that

$$\begin{aligned}\mathbf{E}_x \left[e^{-r\tau^*} g(X_{\tau^*}) \right] &= \int_0^\infty \mathbf{E}_x \left[e^{-r\tau^*} g(X_{\tau^*}) | T_1 = t \right] \lambda e^{-\lambda t} dt \\ &= \mathbf{E}_x \int_0^\infty e^{-rt} \mathbf{E}_{X_t} \left[e^{-r\tau_0^*} g(X_{\tau_0^*}) \right] \lambda e^{-\lambda t} dt \\ &= \mathbf{E}_x \int_0^\infty e^{-rt} V_0(X_t) \lambda e^{-\lambda t} dt \\ &= V(x)\end{aligned}$$

for all $x \in \mathbf{R}_+$. □

Lemma 2.6 ends the sequence of auxiliary results needed to prove the main theorem 1.1. We close the section by studying the asymptotics of the optimal characteristics x^* and V as $\lambda \to 0$ and $\lambda \to \infty$. To this end, denote as $\hat{x}$ the stopping threshold determined (uniquely) by the smooth-pasting condition $g(\hat{x})\psi'(\hat{x}) = \psi(\hat{x})g'(\hat{x})$ and as $\hat{V}$ the value function constituted by this threshold policy – i.e., let

$$\hat{V}(x) = \begin{cases} g(x), & x \geq \hat{x}, \\ \frac{g(\hat{x})}{\psi(\hat{x})}\psi(x), & x \leq \hat{x}. \end{cases} \tag{2.14}$$

Using this notation, we have the following result.

Proposition 2.7. *Let* x^*, V *and* V_0 *be given by Theorem 1.1. Then*

(1) x^* *is an increasing function of* λ
(2) $x^* \to \hat{x}$, $V(x) \to \hat{V}(x)$ *and* $V_0(x) \to \hat{V}(x)$ *as* $\lambda \to \infty$
(3) $V(x) = 0$ *and* $V_0(x) = g(x)$ *when* $\lambda = 0$,

for all $x \in \mathbf{R}_+$.

Proof. We start by noticing that on the limit $\lambda = 0$ the signal process jumps only at $T_0 = 0$ and $T_\infty = \infty$ implying that $V(x) = 0$ and $V_0(x) = g(x)$ for all $x \in \mathbf{R}_+$. Now, let $x \geq \hat{x}$. Since diffusions are Feller processes, we have that $\lambda(R_{r+\lambda}g)(x) \to g(x)$ as $\lambda \to \infty$; in fact, we have convergence even in sup-norm, see [12]. By coupling this with the representation

$$V(x) = \lambda(R_{r+\lambda}g)(x) + (g(x^*) - \lambda(R_{r+\lambda}g)(x^*))\,\mathbf{E}_x\left[e^{-(r+\lambda)\tau_{x^*}}\right]$$

(see (1.8)), we deduce that $V(x) \to g(x)-$ as $\lambda \to \infty$. Monotonicity of this convergence and continuity of V across the boundary x^* imply that x^* increases as λ increases and, consequently, that $x^* \to \hat{x}$ as $\lambda \to \infty$. Finally we conclude that $V(x) \to \hat{V}(x)$ and $V_0(x) \to \hat{V}(x)$ for all $x \in \mathbf{R}_+$ as $\lambda \to \infty$. $\square$

3. Illustrations

3.1. **Geometric Brownian motion.** In this subsection we will analyze the problem studied in [7], namely the perpetual American put option, where the underlying dynamics follow a geometric Brownian motion. Let X be the regular linear diffusion with infinitesimal generator

$$\mathcal{A} = \frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} + \mu x \frac{d}{dx},$$

where $\mu \in \mathbf{R}$ and $\sigma > 0$. The optimal stopping problem can now be formulated as

$$V(x) = \sup_{\tau} \mathbf{E}_x\left[e^{-r\tau}(X_\tau - K)^+\right], \tag{3.1}$$

where $r > 0$ satisfies the condition $\mu < r$ and K is an exogenously given constant. The scale density S' reads as $S'(x) = x^{-\frac{2\mu}{\sigma^2}}$ and the speed density m' reads as $m'(x) = \frac{2}{(\sigma x)^2}x^{\frac{2\mu}{\sigma^2}}$. The functions ψ and φ_λ can now be written as $\psi(x) = x^b$ and $\varphi_\lambda(x) = x^a$, where $b = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1$ and $a = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} < 0$. A simple computation implies that the Wronskian $B_\lambda = 2\sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}}$ and, consequently, that the resolvent $\lambda(R_{r+\lambda}g)$ of the payoff function $g : x \mapsto (x-K)^+$ reads as

$$\lambda(R_{r+\lambda}g)(x) = \frac{\lambda}{r+\lambda-\mu}x - \frac{\lambda}{r+\lambda}K \tag{3.2}$$

for all $x \in \mathbf{R}_+$.

We will now turn to the determination of the optimal stopping threshold x^* and the optimal value function V. First, elementary integration implies that

$$J(x) = \frac{2}{\sigma^2\alpha}x^{-\alpha},$$

where $\alpha = \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0$ for all $x \in \mathbf{R}_+$. Similarly we find that

$$I(x) = \begin{cases} \frac{2}{\sigma^2}x^{-\beta}\left(\frac{x}{\beta-1} - \frac{K}{\beta}\right) & x > K \\ 0 & x < K, \end{cases}$$

where $\beta = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1$. Let $x > K$. Then by elementary computations we find that $\frac{I(x)}{J(x)} = \alpha x^{\alpha-\beta}\left(\frac{x}{\beta-1} - \frac{K}{\beta}\right)$ and

$$\frac{d}{dx}\left(\frac{I(x)}{J(x)}\right) \gtreqqless 0, \quad \text{when } x \lesseqqgtr x^* := \frac{b(\beta-1)}{\beta(b-1)}K.$$

We remark that it is a straightforward computation to verify that

$$\frac{b(\beta-1)}{\beta(b-1)} = \frac{b - \frac{r}{r+\lambda}a}{b - \frac{(r-\mu)a-\lambda}{r+\lambda-\mu}};$$

see [7], p. 147, expression (15). Finally, using the expressions (3.2) and (1.8) we obtain the representation

$$V(x) = \begin{cases} \frac{\lambda}{r+\lambda-\mu}x - \frac{\lambda}{r+\lambda}K + \frac{\frac{r-\mu}{r+\lambda-\mu}x^* - \frac{r}{r+\lambda}K}{\varphi_\lambda(x^*)}\varphi_\lambda(x) & x > x^* \\ \frac{x^*-K}{\psi(x^*)}\psi(x) & x < x^* \end{cases}$$

for the optimal value V; see [7], pp. 146–147, expressions (13), (14) and (16). Thus we have recovered the results on x^* and V by Dupuis and Wang from our analysis. A straightforward differentiation yields

$$\frac{dx^*}{d\lambda} = \hat{x}\frac{1}{\beta^2}\frac{d\beta}{d\lambda} = \hat{x}\left(\beta^2\sigma^2\sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}}\right)^{-1} > 0,$$

this observation is in line with Part (1) of Proposition 2.7. Moreover, since $\beta \to \infty$ as $\lambda \to \infty$, we see immediately from the representation of x^* that $x^* \to \hat{x} := \frac{bK}{b-1}$ as $\lambda \to \infty$. Finally, since $\frac{\varphi_\lambda(x)}{\varphi_\lambda(x^*)} < 1$ whenever $x > x^*$, we find after elementary manipulations that

$$\frac{\lambda}{r+\lambda-\mu}x - \frac{\lambda}{r+\lambda}K + \frac{\frac{r-\mu}{r+\lambda-\mu}x^* - \frac{r}{r+\lambda}K}{\varphi_\lambda(x^*)}\varphi_\lambda(x) \to x - K$$

for all $x > x^*$ and, consequently, that both $V(x)$ and $V_0(x)$ tend to

$$\hat{V}(x) = \begin{cases} x - K & x > \hat{x} \\ \frac{\hat{x}-K}{\psi(\hat{x})}\psi(x) & x > \hat{x} \end{cases}$$

as $\lambda \to \infty$.

To end the subsection, we illustrate graphically in Figure 1 the value functions V, V_0 and $\hat{V}$ under the parameter configuration $\mu = 0.01$, $r = 0.05$, $\sigma^2 = 0.1$, $\lambda = 0.1$ and $K = 1.2$.

3.2. **Logistic diffusion.** As a generalization of the geometric Brownian setting, we consider the case of the optimal stopping problem (3.1) where the state process X follows a regular linear diffusion with infinitesimal generator

$$\mathcal{A} = \frac{1}{2}\sigma^2x^2\frac{d^2}{dx^2} + \mu x(1-\gamma x)\frac{d}{dx},$$

where exogenous constants $\mu, \gamma, \sigma \in \mathbf{R}_+$. This process is called the *logistic diffusion* (or the *geometric Ornstein-Uhlenbeck process* [8] or the *radial Ornstein-Uhlenbeck process* [2]) and was made famous in literature of real options at the latest by [6]. As

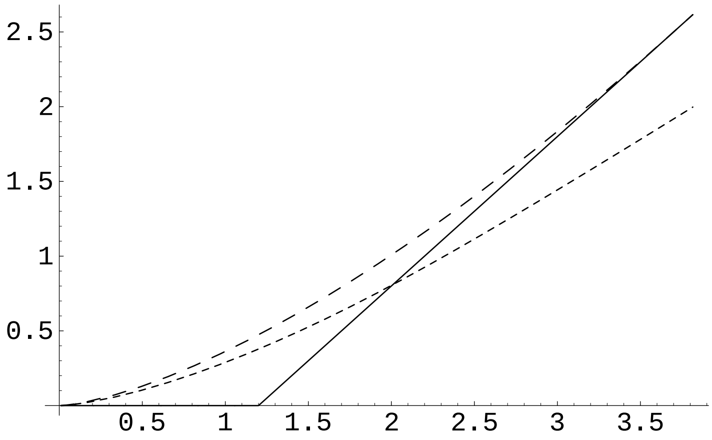

Figure 1. The value functions $\hat{V}$ under the complete information (upper dashed curve) and V under the information rate $\lambda = 0.1$ (lower dashed curve). The solid line is the payoff $g : x \mapsto (x - K)^+$ and the value function V_0 can be recovered from the figure by first following V and then after the intersection the payoff g.

above, a straightforward computation yields the scale density $S'(x) = x^{-\frac{2\mu}{\sigma^2}} e^{\frac{2\gamma\mu}{\sigma^2} x}$ and, consequently, the speed density $m'(x) = \frac{2}{(\sigma x)^2} x^{\frac{2\mu}{\sigma^2}} e^{-\frac{2\gamma\mu}{\sigma^2} x}$ for all $x \in \mathbf{R}_+$.

We know from the literature (e.g., see [5], section 6.5) that the increasing solution ψ and the decreasing solution φ_λ can be expressed as

$$\begin{cases} \psi(x) = x^b M(b, 2b + \frac{2\mu}{\sigma^2}, \frac{2\mu\gamma}{\sigma^2} x) \\ \varphi_\lambda(x) = x^\alpha U(\alpha, 2\alpha + \frac{2\mu}{\sigma^2}, \frac{2\mu\gamma}{\sigma^2} x), \end{cases}$$

where $b = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$ and $\alpha = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}}$. Due to the difficult nature of the functions ψ and φ_λ, we will now fix a parameter setting and illustrate the results numerically. In Table 1 we present the optimal stopping thresholds for different information rates λ under the parameter configuration $\mu = 0.01$, $r = 0.05$, $\sigma^2 = 0.1$, $\gamma = 0.5$, and $K = 1.2$.

λ	0.1	0.5	1	5	10	100	∞
x^*	1.77784	2.26496	2.47876	2.81011	2.8952	3.04044	3.10943

Table 1. The optimal stopping threshold x^* for various information rates λ and smooth pasting boundary $\hat{x}$ under the parameter configuration $\mu = 0.01$, $r = 0.05$, $\sigma^2 = 0.1$, $\gamma = 0.5$, and $K = 1.2$.

The numerical results reported in Table 1 are in line with our main results. In particular, these numerics indicate that the optimal stopping threshold x^* is a increasing function of the information rate λ and the these thresholds tend to the smooth-pasting threshold $\hat{x}$ as λ increases.

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