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ABSTRACT

In this paper we study the number of pure strategy Nash equilibria in large finite *n*-player games. A distinguishing feature of our study is that we allow general - potentially multivalued - best reply correspondences. Given the number *K* of pure strategies to each player, we assign to each player a distribution over the number of his pure best replies against each strategy profile of his opponents. If the means of these distributions have a limit μ_i for each player i as the number *K* of pure strategies goes to infinity, then the limit number of pure equilibria is Poisson distributed with a mean equal to the product of the limit means μ_i. In the special case when all best reply mappings are equally likely, the probability of at least one pure Nash equilibrium approaches one and the expected number of pure Nash equilibria goes to infinity.

JEL Classification: C62, C72

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1 Introduction

To understand how Nash equilibrium behaves as a solution concept, on the "average", is one of the foundational questions of game theory. A important literature has analyzed pure Nash equilibria (PNE) in random games (see e.g. Stanford 1995a,b, 1996; Powers 1990, Goldberg et al. 1968, Dresher 1970). Of particular interest is the asymptotic distribution of PNEs when the size of the game becomes large.

It is clear that the the only thing that matters for the distribution of PNEs in a normal form game is the distribution of the players' best responses in the game. A standard assumption in the random games literature has, however, not to start from best responses but rather to derive best responses from payoffs that are randomly drawn. Crucially, the set of feasible utilitites is conceived to be infinitely times larger than the finite choice set - an assumption that guarantees that the players are never indifferent, i.e., that the best responses are unique. Under this assumption, it is shown that the limit number of PNEs is Poisson distributed with the mean 1 (see e.g. Stanford 1995a,b, 1996; Powers 1990, Goldberg et al. 1968, Dresher 1970).

In this paper, we study equilibrium formation in a framework where the numbers of best responses are drawn independently from a general distribution. This is an important generalization of the standard analysis since, as is also suggested by game theory texts, multiple best responses are merely a norm rather than an exception in applications. In fact, we are not aware of any theoretical or empirical argument that substantiates the assumption that the optimal choice is always unique. In our opinion it is also is natural to think that the number of best responses is at least some degree responsive to the number of distinct choices.

But allowing general best responses is a theoretically challenging exercise. It is clear that multiplicity of best responses fundamentally changes the probabilistic structure of the PNEs. What drives the existing results (e.g Stanford 1995a,b) concerning the distribution of the PNEs is that, at least in the limit, they can be treated as independent Bernoulli variables. A driving reason for this is that, with single valued best responses, a row or a column does not ever contain more than one PNE. Therefore, what one only needs to show for the result is that a likelihood of a PNE in a row is independent of a PNE in another row in the limit. But with multiple best responses, this is no longer sufficient as there may be many PNEs in the same row or column. This second order effect creates a complicated dependence structure between the PNEs, and it is no longer clear how the limit distribution of the number of PNEs behaves.

We characterize the limit distribution of the PNEs when the random game becomes large. By allowing n players and arbitrary random best response correspondences, our approach considerably generalizes the literature. Our main result is that if the *average number* of best responses of the players is bounded, then the limit game can still be approximated by a Poisson distribution: when the game becomes large, the number of PNEs is Poisson distributed with mean $\Pi_i \bar{\mu}_i$, where $\bar{\mu}_i$ is the limit of the average number of best responses of a player $i = 1, ..., n$. In particular, the shape of the distributions where the best responses are drawn does not affect the limit distribution of the PNEs as it is only the average numbers that matter. Thus, if the average numbers of best responses are bounded, then the PNEs can be treated in the limit as if they are independent random variables no matter how complicated the limit best response structures may be.

We also argue that if some of the player's average number of best responses increases without a bound as the game becomes large, which is the case if each best response correspondence is equally likely, then the number PNEs increases without a bound with probability one. Moreover, then the probability of at least one PNE approaches unity. Note that it is sufficient that there is only one player with this property.

It is often useful to think that a player's best responses emerge from maximization of a utility function. To complete the analysis, we study the case where randomness concerns the underlying utilities. When the utility indices are not drawn from an infinite but a finite set, the probability of multiple best responses is strictly positive. The standard result that the distribution of pure Nash equilibria converges to the Poisson distribution with mean 1 (Stanford 1995a,b, 1996; Powers 1990, Goldberg et al. 1968, Dresher 1970) is derived under the hypothesis that the utility indices are drawn from a continuum and it does not hold when the utilities are drawn from a finite set. We let the cardinality of the set of possible utilities of a player i, $m_i(K)$, depend on the size of the game K in such a way that $m_i(K)/K$ approaches real number r_i as K becomes large. Then the probability of multiple best responses does not vanish even in the limit. We show that in such case the incidence of a PNE with maximal payoffs approaches Poisson with mean $\Pi_i r_i$. We show that the limit distribution of pure Nash equilibria converges to Poisson with mean $\Pi_i r_i^{-1}(1 - e^{-r_i^{-1}})^{-1}$. Since this number converges to 1 as r_i tends to infinity, our result can be taken as a generalization of the previous literature.

It is important to note that our limit results cover also the case where the sizes of the choice sets of the players, say $K_1, ..., K_n$, increase with different speed. The only thing that is that the limit ratios $m_i(K_i)/K_i$ are well defined for all players. Alternatively, we could draw the utilities from a same set and vary the sizes of the strategy sets, without affecting the qualitative nature of the results.

There are important omissions. Our focus is specifically restricted to pure strategies and independently drawn payoffs. McLennan (1997) allows mixed strategies and Bade et al. (2007) infinite action sets. Rinott and Scarsini (2000) study the case where players' payoffs are dependent.

The paper is organized as follows. The basic set up is specified in Section

2. The main result is stated and proved in Section 3. Section 4 discusses two cases, when the average number of best responses is not bounded, and when the best responses result from maximization of random utilities.

2 Preliminaries

There are players $N = \{1, ..., n\}$, playing a K^n matrix game. Two games with the same players and the same strategy sets are *best reply equivalent* if they induce the same *best response matrices*, one for each player. Since it is only the best responses that determine players' (pure strategy) behavior, it is safe to take the best response matrices as the the primitive of the model.

Denote a typical action of player i by $a_i \in \{1, ..., K\}$, and a typical action profile of all players by $a = (a_1, ..., a_n) \in \{1, ..., K\}^n$. Also denote by $a_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$ a profile of actions of players all but i. Player i's best response matrix is denoted by

$$x_i = [x_i(a)]_{a \in \{1,...,K\}^n}$$

such that

$$\begin{aligned} & x_i(a) \in \{0, 1\}, \text{ for all } a \in \{1, ..., K\}^n, \\ & \textstyle\sum_{a_i=1}^{K} x_i(a_i, a_{-i}) \geq 1, \text{ for all } a_{-i} \in \{1, ..., K\}^{n-1}. \end{aligned}$$

Denote a profile of all players' best response matrices by $x = (x_1, ..., x_n)$ and a profile of all but i's best response matrices by $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$.

We let x be random. Best responses of an agent are assumed to be independent of the best reponses of the other agents as well as the name of the agent's choice. The primitive of the model is the agent dependent probability distrbutions $p_1^K, ..., p_n^K$ on each the choice set $\{1, ..., K\}$, reflecting the probability $p_i^K(k)$ that agent i has k best responses against an action profile a_{-i} of the other players in a $K-$game. That is, for all a_{-i},

$$p_i^K(k) = \Pr\left\{\textstyle\sum_{a_i} x_i(a_i, a_{-i}) = k\right\}.$$

Then the probability that a subset $S \subseteq \{1, ..., K\}$ with $|S| = k$ of agent i's actions comprises the set of best responses against an action profile a_{-i} of the other players is of the form

$$\binom{K}{k}^{-1} p_i^K(k).$$

Denote the expected number of i's best responses against an action profile of the other players in a $K-$game by

$$\sum_k p_i^K(k) \cdot k = \mu_i^K,$$

and by $\bar{\mu}_i \in \mathbb{R}_+$ the limit expected number of best responses, when it exists

$$\mu_i^K \to_K \bar{\mu}_i. \tag{1}$$

Note that $\bar{\mu}_i$ is also the limit *average* number of best responses, by the law of large numbers.

An action profile $a \in \{1, ..., K\}^n$ forms a *pure Nash equilibrium* (PNE) if (and only if)

$$\prod_{i \in N} x_i(a) = 1.$$

The number of PNEs in a $K-$game given the best response matrices $x = (x_1, ..., x_n)$ is denoted by

$$\pi(x) = \sum_a \prod_i x_i(a).$$

Our main interest is in characterizing the limit properties of the random variable π when K becomes large.

3 Main result

We now state the main result of the paper. The remainder of the section provides a proof for it.

Theorem 1 *Assume that a bounded limit $\bar{\mu}_i$ exists for all $i \in N$. Then the number of PNEs π is Poisson distributed with mean $\Pi_i \bar{\mu}_i$ as K goes to infinity.*

The content of Theorem 1 is that, in the limit, the number of PNEs can be computed as if the probability of one action profile being a PNE is independent of another action profile being a PNE. Our proof strategy is to show that, with probability one, there are no two PNEs in the same row (column) of the limit game, and hence the number of PNEs can be computed on the basis of how many rows (columns) contain a PNE.

Given a best response matrix x_i of player i, denote the proportion of the K^{n-1} action profiles of the other players against which i has k distinct best responses by

$$p(k : x_i) := \frac{\sum_{a_{-i}} I\left\{\sum_{a_i} x_i(a) = k\right\}}{K^{n-1}}, \quad \text{for all } k = 1, ..., K,$$

where $I\{\cdot\}$ is an indicator function. Conversely, denote the proportion of the K actions of i that are best responses to ℓ distinct action profiles of the other players by

$$q(\ell : x_i) := \frac{\sum_{a_i} I\left\{\sum_{a_{-i}} x_i(a) = \ell\right\}}{K}, \quad \text{for all } \ell = 0, ..., K^{n-1}. \tag{2}$$

Given x_i, the average number of i's best responses $\mu_i(x_i)$ is denoted by

$$\mu_i(x_i) := \sum_{k=1}^{K} p\,(k : x_i) \cdot k.$$

Since it does not matter whether one counts the sums of entries of a matrix on the basis of rows or columns,

$$K^{n-1} \sum_{k=1}^{K} p\,(k : x_i) \cdot k = K \sum_{\ell=0}^{K^{n-1}} q\,(\ell : x_i) \cdot \ell,$$

and the average number of best responses $\mu_i(x_i)$ can also be written

$$\mu(x_i) = \frac{\sum_{\ell=0}^{K^{n-1}} q\,(\ell : x_i) \cdot \ell}{K^{n-2}}. \tag{3}$$

Note that

$$E_{x_i}\left[\mu(x_i)\right] = \mu_i^K.$$

Thus

$$E_{x_i}\left[\mu(x_i)\right] \to_K \bar{\mu}_i,$$

whenever the limit $\bar{\mu}_i$ exists.

Denote the number of the other players' action profiles a_{-i} against which a_i is a best response

$$\lambda(a_i, x_i) = \sum_{a_{-i}} x_i(a_i, a_{-i}).$$

Also, denote the number of PNEs that a_n is a component of by

$$\pi(a_i, x) = \sum_{a_{-i}} \prod_{j} x_j(a_i, a_{-i}).$$

Finally, denote the number of PNEs in the $n-1$ player game restricted to the player set $N\backslash\{i\}$ when i chooses a_i by

$$\pi(a_i, x_{-i}) = \sum_{a_{-i}} \prod_{j \in N\backslash\{i\}} x_j(a_i, a_{-i}).$$

Construct a new Bernoulli variable

$$\chi\,(x : k, \ell, a_i) = \begin{cases} 1, & \text{if } \lambda(a_i, x_i) = \ell \text{ and } \pi(a_i, x) = k, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

In other words, for a fixed k and ℓ, $\chi\,(x : k, \ell, a_i) = 1$ if and only if the action a_i of players i is a component of k PNEs. In the probablistic sense this is equivalent to say that for ℓ draws without replacement from a population of size K^{n-1} and initial success probability m/K^{n-1}, where

m is the random number PNEs in the $n-1$ -player game restricted to the player set $N\backslash\{i\}$ under i's choice a_i, results in k successes. This means that, for the given ℓ and m, the probability that an action a_i of player i is a component of k PNEs in the K-game follows the hypergeometric distribution $Hyp\left(\cdot : \ell, K^{n-1}, m/K^{n-1}\right)$.[1] Noting that the random variable m is equivalent to $\pi(a_i, x_{-i})$, the success probability of the Bernoulli variable (4) can be expressed as the expectation

$$E_x\left[\chi\left(x : \ell, k, a_i\right)\right] = E_{x_{-i}}\left[Hyp\left(k : \ell, K^{n-1}, \frac{\pi(a_i, x_{-i})}{K^{n-1}}\right)\right]. \tag{5}$$

Moreover, since the players' best responses are independently drawn, $\pi(a_i, x_{-i})$ and $\pi(a_i', x_{-i})$ are independent random variables whenever $a_i \neq a_i'$. Thus it follows that $\chi\left(\cdot : k, \ell, a_i\right)$ is an independent Bernoulli variable with respect to the defining parameters k, ℓ, and a_i. We collect these observations into the following remark:

Remark 2 $\chi\left(\cdot : k, \ell, a_i\right)$ *is an independent Bernoulli variable with success probability (5), for all triplets* $\left(k, \ell, a_i\right)$.

Given x, the number of PNEs can now be written in the form

$$\pi(x) = \sum_k \sum_\ell \sum_{a_i} k \cdot \chi(x : k, \ell, a_i). \tag{6}$$

By Remark 2, this sum consists of independent Bernoulli variables with known success probabilities. Hence we can use the approximation results from the literature to obtain the limit distribution. Recall the following classical result (see Billingsley 1985, Theorem 23.2.).

Theorem 3 *For* $i = 0, ..., L$, *let* $y_i \in \{0,1\}$ *be an independent Bernoulli variable with success probability* $E_{y_i}[y_i] = \beta_i^L$. *If* $\max_{i\in\{0,...,L\}} \beta_i^L \rightarrow_L 0$, *then* $\lim_L \sum_{i=0}^L y_i$ *is Poisson distributed with the mean* $\lim_L \sum_{i=0}^L \beta_i^L$.

A complication in using Theorem 3 is that the terms in the sum (6) are multiplied by factor k. Therefore, a simple summation of distinct trials does not reflect the desired sum. The next lemma states that this concern is not warranted in the limit.

[1] Where $Hyp(\cdot : \ell, T, m/T)$ denotes the hypergeometric distribution with number of draws ℓ, population size T, and the intial success probability m/T, i.e.,

$$Hyp(k : \ell, K, \frac{m}{T}) = \frac{\binom{m}{k}\binom{T-m}{\ell-k}}{\binom{T}{\ell}}, \quad \text{for all } k.$$

Given the player set N, denote by $\bar{\pi}_N$ the limit of the expected number of PNEs

$$E_x[\pi(x)] \to_K \bar{\pi}_N, \tag{7}$$

whenever the limit exists. Then $\bar{\pi}_{N\setminus\{i\}}$ is the limit expected number of the PNEs in an $n-1$ -game restricted to the player set $N\setminus\{i\}$.

Before we state our first lemma, recall the following fact concerning the hypergeometric distribution:

$$\lim_T T^k \cdot Hyp\left(k : \ell, T, \frac{m}{T}\right) = \ell \cdot m. \tag{8}$$

Lemma 4 *Let a sequence $\{q^K\}$ of probability distributions on $\mathbb{N}$ satisfy*

$$\lim_K \frac{\sum_\ell q^K(\ell) \cdot \ell}{K^{n-2}} = \lambda.$$

Assume that a bounded limit $\bar{\pi}_{N\setminus\{i\}}$ exists. Then, for any $a_i \in \mathbb{N}$,

$$\lim_K K^{(k-1)(n-1)+1} \cdot E_x\left[\sum_\ell q^K(\ell) \cdot \chi(x : k, \ell, a_i)\right] = \lambda \cdot \bar{\pi}_{N\setminus\{i\}}. \tag{9}$$

Proof. Since the limit $\bar{\pi}_{N\setminus\{i\}}$ is well defined, we have, for any $a_i \in \mathbb{N}$,

$$\begin{aligned}
&\lim_K K^{(k-1)(n-1)+1} \cdot E_x\left[\sum_\ell q^K(\ell) \cdot \chi(x : k, \ell, a_i)\right] \\
= \;& \lim_K E_x\left[\frac{\sum_\ell q^K(\ell) \cdot K^{k(n-1)} \cdot Hyp\left(k : \ell, K^{n-1}, \pi(a_i, x_{-i})/K^{n-1}\right)}{K^{n-2}}\right] \\
= \;& \lim_K \left(\frac{\sum_\ell q^K(\ell) \cdot \ell}{K^{n-2}}\right) \cdot \lim_K E_{x_{-i}}[\pi(a_i, x_{-i})] \\
= \;& \lambda \cdot \bar{\pi}_{N\setminus\{i\}},
\end{aligned}$$

where the first equality follows from (5) and the second from (8) and the fact that $m = \pi(a_i, x_{-i})$ and ℓ in (8) are independently distributed. ■

Appealing to Theorem 3 and Lemma 4, we now state the key result.

Lemma 5 *Assume that bounded limits $\bar{\pi}_{N\setminus\{i\}}$ and $\bar{\mu}_i$ exist. Then the limit number of PNEs is Poisson distributed with the mean $\bar{\mu}_i \cdot \bar{\pi}_{N\setminus\{i\}}$.*

Proof. The total number of PNEs is

$$\begin{aligned}
\pi(x) &= \sum_k \sum_\ell \sum_{a_i} k \cdot \chi(x : k, \ell, a_i) \\
&= \sum_k k \sum_{a_i} \chi(x : k, \lambda(a_i, x_i), a_i). \qquad (10)
\end{aligned}$$

By Remark 2, the elements in this sum are independent, each with success probability $E_x \left[\chi\left(x : k, \lambda(a_i, x_i), a_i\right)\right]$.. First we state properties of the limit distrbution:

Claim 1: With probability one, the limit number of the PNEs (10) can be computed as the sum

$$\lim_K \pi(x) = \lim_K \sum_{a_i=1}^{K} \chi\left(x : 1, \lambda(a_i, x_i), a_i\right). \tag{11}$$

Proof: By (3) and Lemma 4, the expected value of the limit for the sum starting from $k = 2$ has the property that

$$\begin{aligned} & \lim_K \sum_{k=2}^{\infty} k \cdot E_x \left[\sum_{a_i=1}^{K} \chi\left(x : k, \lambda(a_i, x_i), a_i\right)\right] \\ = \; & \lim_K \sum_{k=2}^{\infty} k \cdot K \cdot E_x \left[\sum_{\ell=0}^{K^{n-1}} q(\ell : x_i) \cdot \chi\left(x : k, \ell, 1\right)\right] \\ = \; & \lim_K \sum_{k=2}^{\infty} \frac{k \cdot \bar{\mu}_i \cdot \bar{\pi}_{N\setminus\{i\}}}{K^{(k-1)(n-1)}} \\ = \; & \lim_K \frac{K \cdot \bar{\mu}_i \cdot \bar{\pi}_{N\setminus\{i\}}}{K^{n-1}(K-1)^2} \\ = \; & 0. \end{aligned}$$

This ends the proof of Claim 1.

Claim 2: With probability one, the limit number of the PNEs (10) can be computed as the sum

$$\lim_K \pi(x) = \lim_K K \cdot \sum_{\ell=0}^{K^{n-1}} q(\ell : x_i) \cdot \chi\left(x : 1, \ell, 1\right).$$

Proof: Recall that $q(\ell : x_i)$ is the proportion of the K actions of i that are best responses to ℓ distinct action profiles of the other players, and that $\lambda(a_i, x_i)$ is the number of the other players' action profiles a_{-i} against which a_i is a best response. Then, since (11) consists of independent random variables for all a_is, Claim 1 implies that the number of the PNEs can, with probability one, be computed as the sum

$$\begin{aligned} \lim_K \pi(x) &= \lim_K \sum_{a_i=1}^{K} \chi\left(x : 1, \lambda(a_i, x_i), a_i\right) \\ &= \lim_K K \cdot \sum_{\ell=0}^{K^{n-1}} q(\ell : x_i) \cdot \chi\left(x : 1, \ell, 1\right). \end{aligned}$$

This ends the proof of Claim 2.

Claim 3: With probability one, the limit number of the PNEs (10) can be computed as the sum

$$\lim_K \pi(x) = \lim_K K \cdot \sum_{\ell \leq K^{n-3/2}} q(\ell : x_i) \cdot \chi(x : 1, \ell, 1) . \qquad (12)$$

Proof: By Claim 2, and since χ is bounded from above by 1, it suffices to show that

$$\lim_K E_x \left[\sum_{\ell \geq K^{n-3/2}} q(\ell : x_i) \right] = 0. \qquad (13)$$

By (3),

$$E_{x_i}[\mu(x_i)] = E_x \left[\sum_{\ell \leq K^{n-1}} q(\ell : x_i) \cdot \frac{\ell}{K^{n-2}} \right] \to_K \bar{\mu}_i. \qquad (14)$$

On the one hand, $E_{x_i}[\mu(x_i)]$ can decomposed into two nonnegative components

$$E_{x_i}[\mu(x_i)] = E_x \left[\sum_{\ell \leq K^{n-3/2}} q(\ell : x_i) \cdot \frac{\ell}{K^{n-2}} \right] + E_x \left[\sum_{\ell \geq K^{n-3/2}} q(\ell : x_i) \cdot \frac{\ell}{K^{n-2}} \right],$$

and, on the other,

$$\lim_K E_x \left[\sum_{\ell \geq K^{n-3/2}} q(\ell : x_i) \cdot \frac{\ell}{K^{n-2}} \right] \geq \lim_K K^{n-3/2} \cdot E_x \left[\sum_{\ell \geq K^{n-3/2}} q(\ell : x_i) \right].$$

Thus (13) is implied by (14) whenever $n \geq 2$. This ends the proof of Claim 3.

Claim 4: With probability one, the limit number of the PNEs (10) is Poisson distributed with the mean $\bar{\mu}_i \cdot \bar{\pi}_{N \setminus \{i\}}$.

Proof: By Claim 2, it suffices to show that (i) the sum (12) satisfies the conditions laid down in Theorem 3, and that (ii) the expected value of the sum (12) is equal to $\bar{\mu}_i \cdot \bar{\pi}_{N \setminus \{i\}}$.

For (i), it suffices to show that

$$\begin{aligned} \lim_K \max_{\ell \leq K^{n-3/2}} E_x[\chi(x : 1, \ell, 1)] &= \lim_K \max_{\ell \leq K^{n-3/2}} E_{x_{-i}} \left[Hyp\left(1 : \ell, K^{n-1}, \frac{\pi(a_i, x_{-i})}{K^{n-1}}\right) \right] \\ &= \lim_K \max_{\ell \leq K^{n-3/2}} \frac{\ell}{K^{n-1}} \cdot \lim_K E_{x_{-i}}[\pi(a_i, x_{-i})] \\ &= \lim_K \frac{\bar{\pi}_{N \setminus \{i\}}}{K^{1/2}} \\ &= 0, \end{aligned}$$

where the second equality follows from (8) and the fact that $m = \pi(a_i, x_{-i})$ and ℓ in (8) are independently distributed.

For (ii), it follows by Lemma 4 that

$$\begin{aligned} \bar{\pi}_N &= \lim_K K \cdot E_x \left[\sum_{\ell=0}^{K^{n-1}} q(\ell : x_i) \cdot \chi(x : 1, \ell, 1) \right] \\ &= \bar{\mu}_i \cdot \bar{\pi}_{N \backslash \{i\}}. \end{aligned}$$

■

Finally, we argue by induction that $\bar{\pi}_{\{1,...,n\}} = \Pi_{i=1}^n \bar{\mu}_i$, for all n. By Lemma 5, this also proves Theorem 1.

Lemma 6 *Assume that the limit $\bar{\mu}_i$ exists for all $i = 1, ..., n$. Then $\bar{\pi}_{\{1,...,m\}} = \Pi_{i=1}^m \bar{\mu}_i$, for all $m = 1, ..., n$.*

Proof. The initial step: By the definition of PNE and condition (1), the statement holds for $m = 1$.

The inductive step: Let, for any $m = 2, ...,$ $\bar{\pi}_{\{1,...,m-1\}} = \Pi_{i=1}^{n-1} \bar{\mu}_i$. By definition, $\bar{\pi}_{\{1,...,m\}}$ is the expected number of PNEs in an $m-$player limit game. Thus, by Lemma 5, $\bar{\pi}_{\{1,...,m\}} = \Pi_{i=1}^m \bar{\mu}_i$. ■

4 Two cases

4.1 Unbounded average numbers of best responses

In this section we let the average number of best responses of at least one player increase without a bound as the size of the game becomes large. Such situation would materialize, for example, when each best response matrix of a player is equally likely. Note that in the current case, *i.e.*, when the limit of μ_i^K is not well defined, the results of the previous section do not apply.

Theorem 7 *Let there be $i \in N$ such that $\mu_i^K \to_K \infty$. Then the probability that the game possesses at most $k = 0, 1, ...$ PNEs goes to zero as K goes to infinity.*

Proof. For any player j, recall that, given p_j^K, the symmetric probability distribution $f_j^K(S)$ on $2^{\{1,...,K\}} \backslash \{\emptyset\}$ that induces $p_j^K(k)$ is defined by

$$f_j^K(S) = \binom{K}{k}^{-1} p_j^K(k), \text{ if } |S| = k.$$

Let $t \in \mathbb{N}$. Construct a new distribution $f_j^{K,t}$, a truncated version of f_j^K, by

$$f_j^{K,t}(S) = \begin{cases} f_j^K(S) + \frac{1}{K}\sum_{\ell>t} p_j^K(\ell), & \text{if } |S| = 1, \\ f_j^K(S), & \text{if } 1 < |S| \leq t, \\ 0, & \text{if } t < |S|. \end{cases}$$

Distributions $(f_1^K, ..., f_n^K)$ and $(f_1^{K,t}, ..., f_n^{K,t})$ induce probability distributions over best response matrices x and, hence, over the number of PNEs $\pi(x)$. Denote the latter distributions by g^K and $g^{K,t}$, respectively. Since $x_j(a) \geq x'_j(a)$ for all a and for all j implies $\pi(x) \geq \pi(x')$, it follows that g^K first order stochastically dominates $g^{K,t}$ for all K, and for all t.

The corresponding distribution $p_j^{K,t}(k)$ on the number of possible best responses is

$$p_j^{K,t}(k) = \begin{cases} p_j^K(1) + \sum_{\ell>t} p_j^K(\ell), & \text{if } k = 1, \\ p_j^K(k), & \text{if } 1 < k \leq t, \\ 0, & \text{if } t < k. \end{cases} .$$

Then there is $\bar{\mu}_j^t \in \mathbb{R}$ such that $\bar{\mu}_j^t \leq t$ and such that

$$\sum_k p_j^{K,t}(k)k \to_K \bar{\mu}_j^t.$$

By assumption there is a player i such that, when the constraint t is relaxed,

$$\bar{\mu}_i^t \to_t \infty. \tag{15}$$

Denote

$$\textstyle\prod_j \bar{\mu}_j^t = \lambda(t).$$

By (15),

$$\lambda(t) \to_t \infty.$$

Since g^K first order stochastically dominates $g^{K,t}$, it follows that, for all k, for all K, and for all t,

$$\sum_{\ell \leq k} g^K(\ell) \leq \sum_{\ell \leq k} g^{K,t}(\ell). \tag{16}$$

By Theorem 1, for all k and for all t,

$$\sum_{\ell \leq k} g^{K,t}(\ell) \to_K \sum_{\ell \leq k} \frac{\lambda(t)^\ell e^{-\lambda(t)}}{\ell!}. \tag{17}$$

Since $\lambda(t) \to_t \infty$ it follows that

$$\frac{\lambda(t)^\ell e^{-\lambda(t)}}{\ell!} \to_t 0.$$

Since (16) and (17) hold for all t,

$$\sum_{\ell \leq k} g^K(\ell) \to_K 0.$$

■

We end this section by stating two immediate corollaries of the previous result. If there is a player whose average number of best responses increases without a bound when the size of the game becomes large, then:

1. The probability that the game possesses at least one PNE goes to one.

2. The expected number of PNEs goes to infinity.

4.2 Utilities drawn from a finite set

We assume in this section that player i's payoffs are drawn uniformly from an $m-$element set $\{1/m_i(K), ..., (m_i(K)-1)/m_i(K), 1\}$, where $m_i(K) \in \mathbb{N}$. We assume that, for each $i = 1, ..., n$, there is a nonnegative real number r_i such that

$$\frac{m_i(K)}{K} \to_K r_i.$$

First we observe the following lower bound on the number of pure PNEs in the limit game. When payoffs for agent i are taken from the set $\{1/m_i(K), ..., (m_i(K)-1)/m_i(K), 1\}$, the best possible PNE is the one with payoffs $(1, ..., 1)$. We first observe that the distribution of number of such best equilibria is approximately Poisson with mean $1/\Pi_i r_i$ as K becomes large.

Proposition 8 *The number of PNE with the maximal payoffs* $(1, ..., 1)$ *is Poisson distributed with mean* $1/\Pi_i r_i$ *as* K *goes to infinity.*

Proof. The probability that an action profile (x, y) results in payoffs $(1, ..., 1)$ gets arbitrarily close to $K^{-n}\Pi_i r_i$ as K becomes large. The probability of payoffs $(1, ..., 1)$ for a given action profile is independent of the realization of the payoffs for other action profiles. Thus the number of action profiles with payoffs $(1, ..., 1)$ is binomially distributed with success probability $K^{-n}/\Pi_i r_i$. The number of trials is K^n and so the mean of this distribution is $K^n \cdot (K^{-n}/\Pi_i r_i) = 1/\Pi_i r_i$. By the well-known approximation result, the limit distribution is Poisson with mean $1/\Pi_i r_i$. ■

As a corollary of the proposition follows that the probability of at least one PNE with payoffs $(1, ..., 1)$ converges to $1 - e^{-1/\Pi_i r_i}$ as K becomes large. However, for all K there is also a positive probability that a PNE materializes with payoffs strictly lower than 1. As long as $r_i > 0$, this probability does not vanish when K becomes large, and it needs to be taken into account when evaluating the distribution of PNEs.

The proof of the following result, which states the limit of the expected number of i's best responses, is relegated to the appendix.

Lemma 9 *For any* i,

$$\mu_i^K \to_K \frac{1/r_i}{1 - e^{-1/r_i}}.$$

The following corollary is implied by Theorem 1.

Corollary 10 *In the limit, the number of PNEs is Poisson distributed with mean*

$$\prod_{i=1}^{n} \left(\frac{1/r_i}{1 - e^{-1/r_i}} \right).$$

Moreover, since

$$\begin{aligned} \lim_{r \to 0} \frac{1/r}{1 - e^{-1/r}} &= \infty, \text{ and} \\ \lim_{r \to \infty} \frac{1/r}{1 - e^{-1/r}} &= 1, \end{aligned}$$

it follows by Theorem 1 that:

1. When payoffs are drawn from a set that is much (infinitely times) larger than the set of choices, the number of pure PNE is Poisson distributed with mean 1 as the set of choices becomes large (cf. Goldberg et al., 1968; Drescher, 1970; Powers, 1990; Stanford, 1995a,b).

2. When payoffs are drawn from a set that is small relative to the size of the game the expected number of PNEs approaches infinity and the probability of at least one PNE approaches one, a result parallel to Theorem 7.

3. Adding a new player $n + 1$ with $r_{n+1} \in (0, \infty)$ increases the expected number of PNEs. If the parameters $r_1, ..., r_n$ are drawn from a bounded set, then the number of expected PNEs grows exponentially in n.

Note also that the ratio between Poisson mean $1/\Pi_i r_i$ in Proposition 8 - the lower bound of the expected number of equilibria - and the Poisson mean $\Pi_i \bar{\mu}_i$ in Theorem 1, i.e. $1/\Pi_i(1-e^{-1/r_i})$ tends to one when all r_is tend to 0, reflecting the fact that when the set of utility indices is small relative the size of the game, most of the PNE are with maximal payoffs.

A note on the limit game The natural limit game when K becomes large is the one in which all players have $\mathbb{N}$ as their strategy sets. If $m_i(K)$ increases without limit as well, for all i, then the uniform distribution over $\{1/m_i(K), \ldots, (m_i(K)-1)/m_i(K), 1\}$ weakly converges to the uniform distribution over $[0,1]$. Assume indeed that the strategy sets are $\mathbb{N}$ and payoffs to both players and to each strategy pair are *i.i.d.* draws from the uniform distribution over $[0,1]$. In this game there are *no* pure Nash equilibria with probability 1. To see this, note that player $i = 1, ..., n$ gets utility strictly less than 1 from every strategy pair with probability 1. Hence a Nash equilibrium $(a_1, ..., a_n)$ should be such that player, say, i gets equilibrium payoff $y < 1$. But with probability one he gets payoff $x > y$ from some other action $a' \neq a_i$. This is one reason why the limit results are of interest: if there were pure Nash equilibria in the limit game, then such an equilibrium might qualify as an approximate solution to a large but finite matrix game.

A Appendix

Proof of Lemma ??. The probability that the number of action profiles a_{-i} against which i has k best responses is the probability that k actions generate the same payoff v times the probability that all other actions generate lower payoffs, given v. Since the distribution over the set $\{0, 1/m_i(K), \ldots, (m_i(K)-1)/m_i(K), 1\}$ is uniform, we have, under given K,

$$\begin{aligned} E_{x_i}\left[p_i\left(k : x_i\right)\right] &= \sum_{v=1}^{m_i(K)} \binom{K}{k} \left(\frac{1}{m_i(K)}\right)^k \left(\frac{v-1}{m_i(K)}\right)^{K-k} \\ &= \binom{K}{k} \left(\frac{1}{m_i(K)}\right)^k \sum_{x=1}^{m_i(K)} \left(1 - \frac{v}{m_i(K)}\right)^{K-k}, \end{aligned}$$

where the second equality follows by reversing the order of summation. Letting K become large,

$$\begin{aligned} \lim_K E_{x_i}\left[p_i\left(k : x_i\right)\right] &= \lim_K \frac{1}{k!} \left(\frac{K}{m_i(K)}\right)^k \sum_{v=1}^{m_i(K)} \left(1 - \frac{v}{m_i(K)}\right)^{K-k} \\ &= \frac{\sum_{v=1}^{\infty} e^{-v/r_i}}{r_i^k k!} \\ &= \frac{e^{-1/r_i}}{(1-e^{-1/r_i}) r_i^k k!}, \end{aligned}$$

where the second equality follows by taking a component wise limit of the summation. Thus

$$\begin{aligned}
\lim_K \mu_i^K &= \lim_K E_x[\mu_i(x_i)] \\
&= \lim_K E_{x_i}\left[\sum_{k=1}^{m_i(K)} kp_i\,(k : x_i)\right] \\
&= \lim_K \sum_{k=1}^{m_i(K)} kp_i\,(k : x_i) \\
&= \sum_{k=1}^{\infty} \frac{ke^{-1/r_i}}{(1-e^{-1/r_i})r_i^k k!} \\
&= \frac{e^{-1/r_i}/r_i}{1-e^{-1/r_i}} \sum_{k=1}^{\infty} \frac{1}{r_i^{k-1}(k-1)!} \\
&= \frac{1/r_i}{1-e^{-1/r_i}},
\end{aligned}$$

where the final equality follows from noting that $\sum_{n=1}^{\infty}\left[r_i^{n-1}(n-1)!\right]^{-1}$ is a Taylor expansion of e^{1/r_i}. ■

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