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ABSTRACT

Pure strategy Markov perfect equilibria (MPE) in dynamic cake sharing problems are analyzed. Each player chooses under perfect information how much to eat from the current cake and how much to leave to the next period. The left over cake grows according to a given growth function. With linear utilities and strictly concave increasing growth function the only symmetric equilibrium with continuous strategies is the trivial equilibrium in which a player eats the whole cake whenever it is his turn to move. This is quite different than in the corresponding single person decision problem (or at a social optimum) where the cake grows from small initial values towards the steady state. A non-trivial equilibrium with a positive steady state exist in the game. In such an equilibrium strategies cannot be continuous. When utilities are concave and the growth function is linear, a nontrivial MPE with a positive steady state may not exist.

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1. Introduction

Pure strategy Markov perfect equilibria (*MPE*) in dynamic cake sharing problems are analyzed. Each player chooses under perfect information how much to eat from an existing cake and how much to leave to the next period. The left over cake grows according to a given growth function. With linear utilities and strictly concave increasing growth function the only symmetric equilibrium with continuous strategies is the trivial equilibrium in which a player eats the whole cake whenever it is his turn to move. This is quite different than in the corresponding single person decision problem (or at a social optimum) where the cake grows from small initial values towards the steady state. A non-trivial equilibrium with a positive steady state exist in the game. In such an equilibrium strategies cannot be continuous. When utilities are concave and the growth function is linear, a nontrivial *MPE* with a positive steady state may not exist.

The result about continuous symmetric equilibria is rather surprising. For example, Battaglini *et.al* (2011) study symmetric continuous Markov equilibria in a public goods provision game with almost perfect information, and they find that there exists a continuum of equilibria with different steady states. There are monotone equilibria (players' investment are increasing functions of the state), and also equilibria with cycles. In their model growth function is linear and utility functions concave.

The classical papers by Levhari and Mirman (1980) and Fershtman and Nitzan (1991) demonstrate that there can be overexploitation of the resource or too little investment in the public good production. The free rider and underinvestment problem show up very dramatically in the continuous equilibria of this paper, since the stock can never increase. A short explanation is in order.

If there were an MPE in continuous strategies in which the stock is increasing from some initial state, then we can show that actually the stock should be growing at the next state as well, and so on. It follows that the stock is growing to some limit state, and continuity of strategies would make that limit state a steady state. But it turns out that then a player has an incentive either to deviate from that (supposed) steady state back to the increasing path, or, to jump from that increasing path "too early" to the steady state.

While positive steady states cannot exist at continuous symmetric equilibria (Theorem 3, p. 9), such a steady state exists in equilibrium with discontinuous strategies (Theorem 4, p. 11). We show that, when the discount factor is sufficiently high, the steady state level of the stock can be inefficiently high and agents consume too little as compared to the social optimum.

The reason is that when Markov strategies are used, there are less possibilities to punish deviations than when history dependent strategies are used. More precisely, to make it unprofitable for agent i to consume too much at a steady state, it must not be profitable for agent j to jump immediately back to steady state if the current state is just below the steady state. A way to guarantee this is that a deviation from steady state is responded by eating the whole stock. This is credible if the steady state stock is very large and steady state consumption small. But even when discontinuous strategies are allowed, the result holds that in equilibrium the steady state cannot be approached from below by an increasing path (Theorem 1, p. 5).

The paper is organized in the following way. The model is presented in Section 2. Results are stated and proved in Section 3. Alternative formulations of the models are shortly discussed in Section 4.

2. The Model

Let the growth function be $x_{t+1} = f(x_t - c_t), t = 0, 1, 2 \ldots$, where f is a strictly concave increasing function satisfying $f'(0) = \infty, f(0) = 0$, and where the consumption c_t in period t satisfies $0 \leq c_t \leq x_t$. There are two players, $i = 1, 2$. Player 1 chooses every odd period t his consumption level $c_{1,t}$, and player 2 chooses her consumption $c_{2,t}$ every even period t. Both players have a linear utility function $u(c) = c$ over per period consumption levels c, and both players have a discount factor $\delta \in (0, 1)$. Both players maximize the discounted sum of their own consumption.

We look at pure strategy Markov perfect equilibria (MPE) of this game. In our setting, a Markov strategy for player $i = 1, 2$ is a function c_i that to each level x of the stock associates the current consumption $c_i(x), 0 \leq c_i(x) \leq x$. Markov perfect equilibrium is a subgame perfect equilibrium such that players use Markov strategies. A equilibrium path $\{x_t\}_{t \geq 0}$ starting from an initial value x_0 is the path generated by equilibrium strategies. If player i moves at x_0, then $x_1 = f(x_0 - c_i(x_0))$

and for each odd t (even t), $x_{t+1} = f(x_t - c_j(x_t)), j \neq i$ $(x_{t+1} = f(x_t - c_i(x_t)))$, $i, j = 1, 2$.

Example 1. Let us outline the solution of a corresponding single person decision problem. The Bellman equation of such a problem is

$$V(x) = \max_c \left\{c + \delta V\big(f(x - c)\big)\right\} \tag{1}$$

Applying the envelope theorem we get

$$V'(x) = \delta V'\big(f(x - c)\big) f'(x - c) \tag{2}$$

At a steady state, $\bar{x} = f(\bar{x} - \bar{c})$. From equation (2) we get that

$$f'(\bar{x} - \bar{c}) = 1/\delta. \tag{3}$$

Hence $\bar{x} - \bar{c} = (f')^{-1}(1/\delta)$, and so $\bar{x}$ satisfies $\bar{x} = f\big((f')^{-1}(1/\delta)\big)$, and then $\bar{c}$ can be solved from $\bar{x} - \bar{c} = (f')^{-1}(1/\delta)$.

To have a numerical example, we will use square root as the growth function, $f(x - c) = \sqrt{x - c}$, and in that case equation (3) reduces to

$$\sqrt{\bar{x} - \bar{c}} = \bar{x} = \delta/2. \tag{4}$$

The corresponding steady state consumption is

$$\bar{c} = \bar{x}(1 - \bar{x}) = \frac{\delta}{2}\Big(1 - \frac{\delta}{2}\Big) \tag{5}$$

The optimal strategy c is the following (in general as well in square root case, since utility is linear). If $x > \bar{x}$, then $c(x) = x - \bar{x} + \bar{c}$. If $0 < x < \bar{x}$, then $c(x) = 0$ as many periods as it takes for x to grow to $\bar{x}$. (Consumption in the last period in this path is such that the steady state is reached in the following period.) In other words, the *most rapid approach path* is employed. In the special case of the square root growth function, we have

$$c(x) = \begin{cases} x - (\delta/2)^2, & \text{if } x \geq \delta/2; \\ \max\{0, x - (\delta/2)^2\}, & \text{if } x < \delta/2. \end{cases} \tag{6}$$

The game situation is quite different. There is a trivial MPE in which player i eats the whole cake whenever it is his turn to make a choice. The reason is simple: if it is player i's turn and he thinks that player $j \neq i$ will always in the future eat the whole cake, then the only rational choice for i is to eat the whole cake as well.

There exists a nontrivial equilibrium with a positive steady state. But unlike in the single person case, cake can not grow in equilibrium, as will be shown in the next section.

3. Results

Let us prove first that even if an MPE has a positive steady state $\bar{x} > 0$, the equilibrium strategy profile c cannot be such that the steady state could be reached from an initial state $x < \bar{x}$.

Theorem 1. *Let $c = (c_1, c_2)$ be an MPE such that there is a steady state $\bar{x} > 0$. Then there cannot be an initial state $x_0 < \bar{x}$ such that the equilibrium path $\{x_t\}_{t\geq 0}$ converges to $\bar{x}$.*

Proof. Suppose $\bar{x}$ is reached from $x_0 < \bar{x}$ in finite time. Then there is a least T such that $\bar{x} = f(x_T - c_i(x_T))$, $x_T < \bar{x}$. But then at the steady state $\bar{x}$, player j would deviate to a strategy c'_j such that $x_T = f(\bar{x} - c'_j(\bar{x}))$, which would give hime strictly more utility than the steady state consumption $\bar{c}$ that satisfies $\bar{x} = f(\bar{x} - \bar{c})$ (or $\bar{c} = \bar{x} - f^{-1}(\bar{x})$). This is because $c'_j(\bar{x}) > c_j(\bar{x})$, and given the strategy c_i, player j would get discounted utility $c'_j(\bar{x})/(1 - \delta^2)$.

Suppose then $x_t < x_{t+1}$ and $\lim_t x_t = \bar{x}$. Let $v_i(x)$ be the equilibrium value function at state x for player i when the equilibrium $c = (c_1, c_2)$ is played.

Step 1. We study first what kind of restrictions to the equilibrium we get when deviations from the steady state must be unprofitable, and also deviations from the path $\{x_t\}$ *to* the steady state must be unprofitable.

If steady state $\bar{x}$ has been reached, then it must not be profitable for player i to eat $\bar{x} - f^{-1}(x_{t-1})$ so that the next state would be x_{t-1}. We must have

$$\bar{x} - f^{-1}(x_{t-1}) + \delta^2 v_i(x_t) \leq \frac{\bar{x} - f^{-1}(\bar{x})}{1 - \delta^2},$$

which can be simplified to

$$(1 - \delta^2)\delta^2 v_i(x_t) \leq f^{-1}(x_{t-1})(1 - \delta^2) + \bar{x}\delta^2 - f^{-1}(\bar{x}). \tag{7}$$

When the state is x_{t-2} it must not be profitable for i to eat $x_{t-2} - f^{-1}(\bar{x})$ (instead of eating the equilibrium amount $x_{t-2} - f^{-1}(x_{t-1})$) so that the next state would be the steady state $\bar{x}$. We must have

$$x_{t-2} - f^{-1}(\bar{x}) + \delta^2 \left[\frac{\bar{x} - f^{-1}(\bar{x})}{1-\delta^2} \right] \leq x_{t-2} - f^{-1}(x_{t-1}) + \delta^2 v_i(x_t),$$

which can be simplified to

$$f^{-1}(x_{t-1})(1-\delta^2) + \bar{x}\delta^2 - f^{-1}(\bar{x}) \leq (1-\delta^2)\delta^2 v_i(x_t). \tag{8}$$

The inequalities in (7) and (8) must both be equalities, and hence we get

$$(1-\delta^2)\big[\delta^2 v_i(x_t) - f^{-1}(x_{t-1})\big] = \bar{x}\delta^2 - f^{-1}(\bar{x}). \tag{9}$$

Replacing t by $t+2$ equation (9) becomes

$$(1-\delta^2)\big[\delta^2 v_i(x_{t+2}) - f^{-1}(x_{t+1})\big] = \bar{x}\delta^2 - f^{-1}(\bar{x}). \tag{10}$$

Equations (9) and (10) imply

$$\delta^2\big[v_i(x_{t+2}) - v_i(x_t)\big] = f^{-1}(x_{t+1}) - f^{-1}(x_{t-1}). \tag{11}$$

Step 2. We study next what kind of restrictions to the equilibrium we get from the fact that it cannot be profitable for a player to create a cycle $x_t \to x_{t-1} \to x_t$

Given the state x_t, player i could deviate and eat $x_t - f^{-1}(x_{t-1})$ so that the state next period would be x_{t-1}, and the state after two periods would be x_t again. This deviation must not be profitable:

$$\frac{x_t - f^{-1}(x_{t-1})}{1-\delta^2} \leq x_t - f^{-1}(x_{t+1}) + \delta^2 v_i(x_{t+2}), \tag{12}$$

which can be simplified to

$$f^{-1}(x_{t+1}) - f^{-1}(x_{t-1}) \leq \delta^2\big[-x_t + f^{-1}(x_{t+1}) + v_i(x_{t+2}) - \delta^2 v_i(x_{t+2})\big].$$

By (11), since $c_i(x_t) = x_t - f^{-1}(x_{t+1})$, this becomes

$$\delta^2\big[v_i(x_{t+2}) - v_i(x_t)\big] \leq \delta^2\big[-c_i(x_t) - \delta^2 v_i(x_{t+2}) + v_i(x_{t+2})\big]. \tag{13}$$

But since $-c_i(x_t) - \delta^2 v_i(x_{t+2}) = -v_i(x_t)$, the inequalities in (12) and (13) must be equalities. Therefore

$$v_i(x_t) = \frac{x_t - f^{-1}(x_{t-1})}{1-\delta^2}, \quad \text{for all } t. \tag{14}$$

Final Step. From the restrictions obtained in *Step 1* and *Step 2*, we can now conclude that value function must take the form $v_i(x_t) = x_t + B$, for a constant B, and from this we can finally reach a contradiction needed to prove the Lemma.

By inserting $v_i(x_t)$ from (14) into the equation (9) and simplifying we get

$$\delta^2 x_t - f^{-1}(x_{t-1}) = \bar{x}\delta^2 - f^{-1}(\bar{x}). \tag{15}$$

By replacing t in (15) by $t+2$, we get

$$\delta^2 x_{t+2} - f^{-1}(x_{t+1}) = \bar{x}\delta^2 - f^{-1}(\bar{x}). \tag{16}$$

Equations (15) and (16) imply

$$\delta^2 \big[x_{t+2} - x_t\big] = f^{-1}(x_{t+1}) - f^{-1}(x_{t-1}),$$

and using equation (11) this becomes

$$v_i(x_{t+2}) - v_i(x_t) = x_{t+2} - x_t. \tag{17}$$

Let x_0 be any initial value such that the increasing equilibrium path $\{x_t\}$ converges to the steady state $\bar{x}$ and player i has the move at x_0. Then by induction we get from (17) that

$$v_i(x_{2t}) = x_{2t} + \big[v_i(x_0) - x_0\big], \; \forall t = 1, 2 \cdots \tag{18}$$

Denoting $B = \big[v_i(x_0) - x_0\big]$ we get

$$v(x_{2t}) = x_{2t} + B = x_{2t} - f^{-1}(x_{2t+1}) + \delta^2\big[x_{2t+2} + B\big], \tag{19}$$

since $c_i(x_{2t}) = x_{2t} - f^{-1}(x_{2t+1})$. This implies

$$(1 - \delta^2)B = \delta^2 x_{2t+2} - f^{-1}(x_{2t+1}) \tag{20}$$

Hence by equation (16),

$$B = \frac{\bar{x}\delta^2 - f^{-1}(\bar{x})}{1 - \delta^2}. \tag{21}$$

Since (7) must hold as equality, and $v_i(x_t) = x_t + B$, we have

$$\bar{x} - f^{-1}(x_{t-1}) + \delta^2\big[x_t + B\big] = x_t + B,$$

from which we can solve B:

$$B = \frac{\delta^2 x_t - f^{-1}(x_{t-1})}{1 - \delta^2} + \frac{\bar{x} - x_t}{1 - \delta^2}. \tag{22}$$

Equations (15), (21) and (22) impy that $\bar{x} = x_t$, a contradiction. $\square$

A strategy profile $c = (c_1, c_2)$ is *symmetric*, if $c_1(x) = c_2(x)$ for all x. The next result says that there cannot exists a symmetric MPE with continuous strategies c_i such that in equilibrium $x_1 = f(x_0 - c_i(x_0)) > x_0$ for some initial value x_0. Hence the stock cannot be increasing in such an equilibrium.

Theorem 2. *If (c_1, c_2) is a symmetric MPE with continuous strategies c_i, then $x_1 = f(x_0 - c_i(x_0)) \leq x_0$ for all initial values x_0.*

Proof. Let $v_i(x)$ the the equilibrium payoff of player i when he has the move and the current state is x. Since (c_1, c_2) is a symmetric equilibrium, we have $v_1 = v_2 = v$ and $c_1 = c_2$. Let us abuse notation slightly in this proof and denote $c = c_1 = c_2$. Suppose that $x_0 < x_1 = f(x_0 - c(x_0))$ for some x_0. Using equilibrium strategy at x_0 must be at least as profitable as eating the amount $x_0 - f^{-1}(x_0)$ now so that the opponent would have to eat $c(x_0)$ at state x_0 next period. That is,

$$v(x_0) \geq x_0 - f^{-1}(x_0) + \delta^2 v(x_1). \tag{23}$$

The equilibrium value $v(x_0)$ satisfies

$$v(x_0) = x_0 - f^{-1}(x_1) + \delta^2 v(f(x_1 - c(x_1))). \tag{24}$$

Equations (23) and (24) imply that

$$f^{-1}(x_0) - f^{-1}(x_1) + \delta^2 v(f(x_1 - c(x_1))) \geq \delta^2 v(x_1). \tag{25}$$

Since $f^{-1}(x_0) - f^{-1}(x_1) < 0$, and $f(x_1 - c(x_1)) = x_2$, we get that

$$v(x_2) = v(f(x_1 - c(x_1))) > v(x_1). \tag{26}$$

But this means that $x_2 > x_1$, since the value function v is strictly increasing as is easily shown.

By induction, $x_{t+1} > x_t$ for all $t \geq 0$. The increasing sequence $\{x_t\}$ is bounded above by the fixed point x^* of f. Since f is strictly concave and increasing and $f'(0) > 1 > 0 = f(0)$, a unique fixed point $x^* > 0$ exists. Hence the sequence $\{x_t\}$ has a limit, say $\hat{x}$.

Since equilibrium strategy c is continuous, we have that

$$\lim_{t \to \infty} c(x_t) = c(\hat{x}) \equiv \hat{c}.$$

Since $|x_{t+1} - f(x_t - c(x_t))| \longrightarrow 0$ as $t \longrightarrow \infty$, $\hat{x}$ must be a steady state: $\hat{x} = f(\hat{x} - \hat{c})$. But then $\{x_t\}$ is an increasing sequence converging to a steady state, a contradiction with Theorem 1. □

Theorem 3. *If (c_1, c_2) is a symmetric MPE with continuous strategies c_i, then there are no positive steady states $\bar{x} > 0$ and $c_i(x) = x$ for all x.*

Proof. Suppose that there is a positive steady state $\bar{x} > 0$.

Assume first that there is another steady state $\bar{y} = \bar{x} - \varepsilon < \bar{x}$ such that $\bar{x} \leq f(\bar{x} - \varepsilon)$. That is if at $\bar{y}$ an agent does not eat anything then the next period state is at least $\bar{x}$. Since both steady states are reachable from the other steady state by one step, necessary equilibrium conditions at these steady states are:

$$\frac{\bar{x}-\varepsilon-f^{-1}(\bar{x}-\varepsilon)}{1-\delta^2} \geq \bar{x}-\varepsilon-f^{-1}(\bar{x})+\delta^2\left[\frac{\bar{x}-f^{-1}(\bar{x})}{1-\delta^2}\right], \tag{27}$$

$$\frac{\bar{x}-f^{-1}(\bar{x})}{1-\delta^2} \geq \bar{x}-f^{-1}(\bar{x}-\varepsilon)+\delta^2\left[\frac{\bar{x}-\varepsilon-f^{-1}(\bar{x}-\varepsilon)}{1-\delta^2}\right]. \tag{28}$$

Equation (27) states that it must not be profitable to jump from the steady state $\bar{y}=\bar{x}-\varepsilon$ to $\bar{x}$, and equation (28) says that it must not be profitable to jump from steady state $\bar{x}$ to $\bar{y}=\bar{x}-\varepsilon$. Solving the system (27)-(28) gives us

$$f^{-1}(\bar{x})-f^{-1}(\bar{x}-\varepsilon)=\delta^2\varepsilon. \tag{29}$$

Denote $\bar{c}=c(\bar{x})$, the consumption at the steady state $\bar{x}$, and let $c(\bar{x}-\varepsilon)$ be the consumption at the steady state $\bar{x}-\varepsilon$. Using steady state equations, we can rewrite (29) as

$$\bar{x}-\bar{c}-(\bar{x}-\varepsilon)+c(\bar{x}-\varepsilon)=\delta^2\varepsilon,$$

from which we can solve

$$c(\bar{x}-\varepsilon)=\bar{c}-\varepsilon(1-\delta^2). \tag{30}$$

Then at the steady state $\bar{x}-\varepsilon$ we must have

$$\bar{x}-\varepsilon=f(\bar{x}-\varepsilon-\bar{c}+\varepsilon(1-\delta^2))=f(\bar{x}-\bar{c}-\varepsilon\delta^2),$$

a contradiction since $\varepsilon>0$.

Therefore, if there are two steady states $\bar{x}$ and y, $\bar{y}<\bar{x}$, then $f(\bar{y})<\bar{x}$ and $\bar{x}$ cannot be reached from $\bar{y}$ at one step. It follows that any decreasing sequence $\{\bar{x}_t\}_t$ of positive steady states $\bar{x}_t>0$ must be finite. So there is a least positive steady state $\bar{y}>0$.

If $0<x_0<\bar{y}$, then $x_1<f(x_0-c(x_0))$. Now there cannot be equilibrium path $x_0>x_1>x_2>x_3>0$ since then the player who eats at x_0 the amount $c(x_0)=x_0-f^{-1}(x_1)$ has a profitable deviation $c'=x_0-f^{-1}(x_3)$. It follows that $c(x_0)=x_0$ must hold at each $x_0<\bar{y}$.

But then c cannot be continuous at $\bar{y}$. This contradiction implies that there cannot exist any steady state $\bar{x_0}>0$. By Theorem 2, $x_1<f(x_0-c(x_0))$ must hold at every initial value x_0, and hence by the previous paragraph $c(x)=x$ for all x. $\square$

If symmetry and continuity assumptions are relaxed, we do have nontrivial equilibrium with a positive steady state $\bar{x} > 0$.

Theorem 4. *There is an MPE with a steady state stock $\bar{x} > 0$ satisfying $\delta^2\bar{x} - f^{-1}(\bar{x}) = 0$. The equilibrium strategies $c_i, i = 1, 2$ are*

$$c_i(x) = \begin{cases} x - f^{-1}(\bar{x}), & \text{if } x \geq \bar{x}; \\ x, & \text{if } x < \bar{x}. \end{cases}$$

Proof. The proposed equilibrium strategies have the following property. If the cake is at least as great as the steady state level $\bar{x}$, a player eats so much that steady state is reached in the next period. If the cake is smaller than the steady state level, then a player eats the whole cake. Hence there is no growth in this equilibrium. Let us verify that the strategies given above indeed form an MPE.

Note that at a steady state, $\bar{x} = f(\bar{x} - \bar{c})$, where $\bar{c} = c_i(\bar{x})$, and hence $\bar{c} = \bar{x} - f^{-1}(\bar{x})$. Hence $\bar{x}$ can be sustained as a steady state in equilibrium *only if*

$$\bar{x} \leq \frac{\bar{x} - f^{-1}(\bar{x})}{1 - \delta^2}. \tag{31}$$

This must hold because the right hand side of this inequality is the discounted sum of consumption at a steady state, and the left hand side is the utility if the cake $\bar{x}$ is eaten at once. Equation (31) implies that

$$\delta^2\bar{x} - f^{-1}(\bar{x}) \geq 0. \tag{32}$$

Actually (32) holds as equality,

$$\delta^2\bar{x} - f^{-1}(\bar{x}) = 0. \tag{33}$$

instead of just satisfying the inequality.

To see this, note that if (32) holds as a strict inequality, and i would observe the current state x is only a little bit smaller than $\bar{x}$, then i could choose c so that steady state would be reached in the next period. That is, $c = x - f^{-1}(\bar{x})$. Since j is expected to stay at the steady state if steady state is reached, this would give i a strictly larger discounted payoff than eating the whole cake x immediately, because

$$x - f^{-1}(\bar{x}) + \frac{\delta^2\left[\bar{x} - f^{-1}(\bar{x})\right]}{1 - \delta^2} > x, \quad \textit{iff} \\ \delta^2\bar{x} > f^{-1}(\bar{x}).$$

Therefore (33) must hold when the strategies given in the Theorem are used. □

In the special case when the growth function is the square root, we have the following.

Corollary 1. *If $f(x-c) = \sqrt{x-c}$ then the steady state of Theorem 1 is $\bar{x} = \delta^2$, and the equilibrium strategies are*

$$c_i(x) = \begin{cases} \delta^2(1-\delta^2), & \textit{if } x \geq \delta^2 \\ x, & \textit{if } x < \delta^2. \end{cases}$$

Remark 1. Equilibrium strategies in Theorem 4 and Corollary 1 are neither continuous nor monotone. Discontinuity takes place at the steady state $\bar{x}$. Nonmonotonicity: $c_i(x) = x > c_i(\bar{x})$ when $x < \bar{x}$ is very close to $\bar{x}$, but $c_i(x) = x < c_i(\bar{x})$ when $x < \bar{x}$ is very close to 0.

Remark 2. Equilibria in Theorem 4 and Corollary 1 are symmetric, but symmetry was a result not an assumption.

Remark 3. The equilibrium value functions v_i satisfy $v_i(x) = x$. This follows immediately when $x < \bar{x}$. When $x \geq \bar{x}$, $v_(x) = x$ holds since staying at the steady state $\bar{x}$ gives as much utility as eating the cake $\bar{x}$ immediately.

Remark 4. Since utilities are linear, the social optimum would correspond to the problem where the sum of players' utilities is the objective function. The solution is the one found in Example 1. Note that steady state $\bar{x} = \delta^2$ in the game situation is larger than the steady state $\delta/2$ at the social optimum, iff $\delta > 1/2$.

Remark 5. The unique equilibrium in the finite horizon version of the game is the trivial equilibrium: $c_i(x) = x$ for all x.

4. Variations of the Model

4.1. **Model 1.1.** Suppose players move during the same period under perfect information so that player 1 moves first. Assume also that the time interval between choices is so short that the cake cannot grow during that time. There is no MPE with a positive steady state. To see this, suppose that $\bar{x} > 0$ is a steady in equilibrium $c = (c_1, c_2)$. Then

$$\bar{x} = f\big(\bar{x} - c_1(\bar{x}) - c_2(\bar{x} - c_1(\bar{x}))\big)$$

If $c_1(\bar{x}) = \bar{x} - f^{-1}(\bar{x})$, then $c_2(\bar{x} - c_1(\bar{x})) = 0$. But then player 2 would deviate and choose $c_2(x) = x$ when the size of cake is x and it is his turn to choose. So there cannot exist and MPE with a positive steady state.

4.2. **Model 1.2.** Players move in the same period under perfect information but the first mover is chosen by flipping a fair coin. Now there is an MPE with a steady state $\bar{x} > 0$. At a steady state, the first mover eats the whole surplus $\bar{x} - f^{-1}(\bar{x})$ and the second mover gets zero. If $x > \bar{x}$, then the first mover eats $x - f^{-1}(\bar{x})$ and the second mover gets zero so that steady state is maintained. If $x < \bar{x}$, then whoever has the move, eats the whole cake. A necessary condition for these strategies to form an MPE is that the second mover doesn't want to deviate at the steady state by eating the whole residual cake $f^{-1}(\bar{x})$. That is, we must have

$$\frac{1}{2}\left[\frac{\delta(\bar{x} - f^{-1}(\bar{x}))}{1-\delta}\right] \geq f^{-1}(\bar{x}). \tag{34}$$

This holds, if

$$\delta\bar{x} \geq (2-\delta) f^{-1}(\bar{x}). \tag{35}$$

At the greatest steady state $\bar{x}$, (35) and (36) must hold as equalities. In the case $f(x-c) = \sqrt{x-c}$, the steady state will be

$$\bar{x} = \frac{\delta}{2-\delta}. \tag{36}$$

4.3. **Model 1.3.** Assume the move structure is the same as in the original model, but that utility functions u is strictly concave with $u(0) = 0$ and $u'(0) = \infty$. The growth function is linear: $x_{t+1} = (1+r)(x_t - c_t)$. We assume that the growth rate is the inverse of the discount factor, so $\delta = 1/(1+r)$.

It can be shown that in the single person case any positive initial state $x_0 > 0$ is also a steady state. The steady state consumption level is $c = x_0(1-\delta)$.

A necessary condition for a positive steady state $\bar{x} > 0$ in the two-person game is that it is not profitable to eat the (steady state) cake at once. This holds if

$$u(\bar{x}(1-\delta)) \geq u(\bar{x})(1-\delta^2). \tag{37}$$

For example, if $u(x) = x^\alpha, 0 < \alpha < 1$, then equation (37) implies that

$$\frac{(1-\delta)^\alpha}{1-\delta^2} \geq \bar{x}^{1-\alpha}. \tag{38}$$

If $\delta = 0.9$ and $\alpha < 0.72$, then (38) cannot be satisfied, so there cannot be any MPE with a positive steady state.

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