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ABSTRACT

We study games with utilitarian preferences: the sum of individual utility functions is a generalized ordinal potential for the game. It turns out that generically, any finite game with a potential, ordinal potential, or generalized ordinal potential is better reply equivalent to a game with utilitarian preferences. It follows that generically, finite games with a generalized ordinal potential are better reply equivalent to potential games. For infinite games we show that a continuous game has a continuous ordinal potential, iff there is a better reply equivalent continuous game with utilitarian preferences. For such games we show that best reply improvement paths can be used to approximate equilibria arbitrarily closely.

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1. Introduction

In potential and ordinal potential games there is a real valued function on the set of strategy profiles that reveals completely the better reply structure of these games. This kind a potential or ordinal potential function is like an aggregate utility function of the society. We study a special case with *utilitarian preferences*: the sum of individual utility functions is a generalized ordinal potential for the game. We show that generically, any finite game with a potential, ordinal potential, or generalized ordinal potential is better reply equivalent to a game with utilitarian preferences. It follows that, generically, finite games with a generalized ordinal potential are better reply equivalent to potential games.

Finite games with a generalized ordinal potential have the property that all (better reply) improvement paths end at a pure strategy Nash equilibrium. A game is then said to have the finite improvement property (FIP). Improvement paths are such that one player at a time changes his current strategy to a strictly better strategy. If the deviating player chooses always a best reply, then the path is called a best reply improvement path.

We show that all best reply improvement paths of a game end at an equilibrium, if and only if there is a better reply equivalent game with *utilitarian best replies*. A game has utilitarian best replies, if the sum of individual utility functions completely reveals the best reply structure of the game. Hence if a game has utilitarian preferences then it also has utilitarian best replies, but not necessarily *vice versa*.

For infinite games we show that a continuous game has a continuous ordinal potential, if and only if there is a better reply equivalent continuous game with utilitarian preferences. For such games we show that best reply improvement paths can be used to approximate equilibria arbitrarily closely in the following sense. Given any strategy profile, there is a best reply improvement path such that either the path ends at an equilibrium, or, all but finitely many first elements of the path are arbitrarily close to the set of pure Nash equilibria.

Therefore in applications such as congestion problems (see Milchtaich

1996; Rosenthal 1973) or certain oligopoly games (see Kukushkin 2004), it shouldn't matter too much what kind of potentials (exact, ordinal or generalized ordinal) are studied. Computationally utilitarian welfare function is quite simple. If one only wants to find one pure Nash equilibrium, then in many instances it would be enough to maximize the utilitarian welfare function. Even if this doesn't work directly in a problem at hand, it may be potentially useful to know that at least there is a better reply equivalent problem where this method works perfectly.

Rosenthal (1973) presented the first well-known congestion model and showed how a pure equilibrium can be found by solving a linear programming problem that actually is a potential maximization problem. Monderer and Shapley (1996) introduced the concepts of (exact) potential, ordinal potential, and generalized ordinal potential for noncooperative games. They showed that a finite game has the FIP, if and only if the game has a generalized ordinal potential. Voorneveld *et.al* (1999) showed that every (exact) potential game is isomorphic to a congestion game (see also Facchini *et.al* 1997).

Milchtaich (1996) introduced and analyzed slightly different class of finite congestion models, called crowding games. He showed that although these games need not have the FIP, they nevertheless have the property that from every strategy profile there is some improvement path ending at an equilibrium. Such games are called weakly acyclic, or they are said to have the weak FIP. Apt and Simon (2012) study various algorithms that can be used to find equilibria in this class of games.

The paper is organized in the following way. In Section 2 we define most of the terms and introduce notation. Main results are stated and proven in Section 3. In subsection 3.1 we show that finite ordinal potential games are better reply equivalent to games with utilitarian preferences. In subsection 3.2 we study the relations between weakly acyclic games and games with utilitarian preferences. In subsection 3.3 the main results for infinite games are given.

2. The model

A normal form game $G = \{N; (A_i)_i; (u_i)_i\}$ specifies a finite player set $N = \{1, \ldots, n\}$, a strategy set A_i of each player $i \in N$, and a utility function $u_i : A \longrightarrow \mathbb{R}$ for each player $i \in N$, where $A = A_1 \times \cdots \times A_n$ is the set of strategy profiles. A game G is finite, if all the strategy sets A_i are finite.

A pure Nash equilibrium of a game G is a strategy profile a such that

$$u_i(a_i, a_{-i}) \geq u_i(\tilde{a}_i, a_{-i}), \forall \tilde{a}_i \in A_i, \forall i \in N. \tag{1}$$

Let $B_i(a) = \{b_i \mid u_i(b_i, a_{-i}) \geq u_i(\tilde{a}_i, a_{-i}), \forall \tilde{a}_i \in A_i\}$ be the set of best replies of player i against the strategy profile i. Denote by $B(a) = \prod_{i \in N} B_i(a)$ the value of the best reply correspondence at a. So a is a pure Nash equilibrium, iff $a \in B(a)$.

Monderer and Shapley (1996) call a sequence $(a^0, a^1, \ldots)$ of strategy profiles $a^t \in A$ an *improvement path* if one player at a time chooses a *better reply* against other players strategies. That is, for some $i \in N$, $u_i(a^{t+1}) = u_i(a_i^{t+1}, a_{-i}^t) > u_i(a^t)$. We may sometimes denote a step in an improvement path by $a^t \to a^{t+1}$. In their paper, a game has a *finite improvement property* (FIP), if every improvement path is finite. This means that every maximal improvement path has a pure Nash equilibrium as its last element. We say that a game G is *acyclic*, if there is no improvement path $(a^0, a^1, \ldots)$ such that $a^t = a^0$ for some $t > 0$. A finite game is acyclic, iff it has the FIP. Acyclicity is a weaker property than FIP in the context of infinite games.

They show (Monderer and Shapley 1996, Lemma 2.5) that a game G has the FIP, if and only if G has a *generalized ordinal potential* (*ordinal potential* in generic games). A generalized ordinal potential for a game G is a function $P : A \longrightarrow \mathbb{R}$ such that $\forall i \in N, a_i, b_i \in A_i, a \in A$ the following holds

$$u_i(a_i, a_{-i}) < u_i(b_i, a_{-i}) \Longrightarrow P(a_i, a_{-i}) < P(b_i, a_{-i}). \tag{2}$$

Such a function is an ordinal potential if $\Longrightarrow$ can be replaced by $\Longleftrightarrow$. A function P is a potential for G, if for all $i, a_i, b_i \in A_i$, and for $a \in A$:

$$u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}) = P(a_i, a_{-i}) - P(b_i, a_{-i}). \tag{3}$$

A game G is called a potential (ordinal potential) game, if it has a potential (ordinal potential). Games that do not necessarily have a generalized ordinal potential may still have the property that from every strategy profile a^0 there exists an improvement path ending at an equilibrium. We say that such games have the *weak* FIP (Friedman and Mezzetti 2001, Kukushkin 2011). Such games are sometimes called *weakly acyclic* (see Milchtaich 1996), especially in the context of finite games where these concepts are equivalent. In infinite games weak acyclicity means that from any initial profile a^0 there exists an improvement path without cycles, but the game need not have the FIP.

Following Apt and Simon (2012) an improvement path $(a^0, a^1, \ldots)$ is called a *best reply improvement path* or (BR-improvement path) if at each stage the player who deviates chooses a best reply. We say that a game G has the *finite best reply property* FBRP, if every BR-improvement path is finite. Further, we say that a game G has the *weak FBRP*, if from every strategy profile a^0 there exists a BR-improvement path ending at an equilibrium.

The term BR-acyclic means that no BR-improvement path has cycles, and weak BR-acyclic means that for any a^0 there is at least one BR-improvement path without cycles. In finite games the terms (weak) BR-acyclic and (weak) FBRP are equivalent.

A game G is a pure coordination game if $u_i = u_j, \forall i, j \in N$, that is, players have identical preferences over strategy profiles. A game G is a zero-sum game if $\sum_i u_i(a) = 0, \forall a \in A$.

Given any games G^1 and G^2 with the same player set N and strategy sets $A_i, i = 1, \ldots, n$, define the sum game G^1+G^2 so that the utility function of player $i \in N$ is the sum $u_i^1 + u_i^2$ of his utility functions in the component games G^1 and G^2.

Every n -person game with strategy sets $A_i, i = 1, \ldots, n$ can be represented as a sum of a zero-sum game and a pure coordination game as follows. Take a game $G = \{N; (A_i)_i; (u_i)_i\}$. To any strategy profile $a \in A$, define $\bar{u}(a) = [\sum_i u_i(a)]/n$. Let G^c be a pure coordination game such that $G^c = \{N; (A_i)_i; (\bar{u})_i\}$. That is, the common utility function of players in G^c

is $\bar{u}$. Let $G^0 = \{N; (A_i)_i; (\underline{u})_i\}$ be a zero sum game such that $\underline{u}_i = u_i - \bar{u}$, and note that $G = G^0 + G^c$. One can show that coordination games and zero-sum games are orthogonal subspaces of the inner product space of all finite games given N and A, and so the decomposition $G = G^0 + G^c$ is unique.

Games $G = \{N; (A_i)_i; (u_i)_i\}$ and $G' = \{N; (A_i)_i; (u'_i)_i\}$ are *better reply equivalent*, if for all $i \in N$ and $a \in A$, $u_i(b_i, a_{-i}) > u_i(a)$ iff $u'_i(b_i, a_{-i}) > u'_i(a)$. Games $G = \{N; (A_i)_i; (u_i)_i\}$ and $G' = \{N; (A_i)_i; (u'_i)_i\}$ are *best reply equivalent*, if for all $i \in N$ their best replies B_i and B'_i, respectively, are the same. Better reply equivalence implies best reply equivalence, but not *vice versa.*.

Definition 1. A game $G = \{N; (A_i)_i; (u_i)_i\}$ has utilitarian preferences, if $u_i(b_i, a_{-i}) - u_i(a_i, a_{-i}) > 0$ implies $\sum_{j \in N}[u_j(b_i, a_{-i}) - u_j(a_i, a_{-i})] > 0$, for all $i \in N, b_i \in A_i$ and $a \in A$. If this holds for best replies $b_i \in B_i(a)$ (but not necessarily for other strategies), then we say that G has utilitarian best replies.

So a game G has utilitarian preferences, iff the utilitarian welfare function is a generalized ordinal potential for G.

We say that property P holds generically, or P holds for generic games, if property P holds in an open dense subset of the (properly topologized) set of games under study. For example, the the subset of games $G = \{N; (A_i)_i; (u_i)_i\}$ such that no player is ever indifferent between two different strategy profiles forms an open dense subset of finite games.

3. Results

In this section we will study the relationship of ordinal potential games and weakly acyclic games to a class of games that have utilitarian best replies or utilitarian preferences. All games in subsection 3.1 and 3.2 are finite. Infinite games are studied in subsection 3.3.

3.1. Games with FIP

Theorem 1. *A game $G = \{N; (A_i)_i; (u_i)_i\}$ has an ordinal potential, iff there is a better reply equivalent game $G' = \{N; (A_i)_i; (u'_i)_i\}$ that has utilitarian preferences.*

Proof. Let P be an ordinal potential for $G = \{N; (A_i)_i; (u_i)_i\}$. Let $G' = \{N; (A_i)_i; (u'_i)_i\}$ be such that $u'_i = P$. Then G and G' are better reply equivalent by the definition of the ordinal potential. Since $\sum_j u'_j(a) = nP(a)$ for all $a \in A$, the game G' has utilitarian preferences.

If $G = \{N; (A_i)_i; (u_i)_i\}$ has utilitarian preferences, then define $P(a) = \sum_j u_j(a)$ for al $a \in A$, and note that P is an ordinal potential for G by Definition 1. □

Note that for generic games $G = \{N; (A_i)_i; (u_i)_i\}$, $u_i(a) \neq u_i(b)$ and also $\sum_j u_j(a) \neq \sum_j u_j(b)$ if $a \neq b$. In particular, a generalized ordinal potential is generically an ordinal potential. This gives us the following.

Corollary 1. *Generically a game $G = \{N; (A_i)_i; (u_i)_i\}$ has FIP, iff there is a better reply equivalent game $G' = \{N; (A_i)_i; (u'_i)_i\}$ that has utilitarian preferences.*

Theorem 1 implies of course that if a game has a potential, then it is better reply equivalent to some game with utilitarian preferences. Generically the converse holds as well.

Theorem 2. *If a game $G = \{N; (A_i)_i; (u_i)_i\}$ has a potential, then G is better reply equivalent to some game $G' = \{N; (A_i)_i; (u'_i)_i\}$ with utilitarian preferences. Generically a game $G = \{N; (A_i)_i; (u_i)_i\}$ with utilitarian preferences is better reply equivalent to some potential game $G' = \{N; (A_i)_i; (u'_i)_i\}$.*

Proof. Suppose $G = \{N; (A_i)_i; (u_i)_i\}$ has a potential P. Then by Theorem 1 the game G is better reply equivalent to a game $G' = \{N; (A_i)_i; (u'_i)_i\}$ with utilitarian preferences.

Suppose $G = \{N; (A_i)_i; (u_i)_i\}$ has utilitarian preferences. The game G can be expressed as a sum $G = G^0 + G^c$. The utility functions of all players

i in the pure coordination game G^c are of the form $\bar{u}_i = (\sum_j u_j)/n$. By the assumption of utilitarian preferences, $u_i(b_i, a_{-i}) - u_i(a_i, a_{-i}) > 0$ implies $\sum_{j \in N}[u_j(b_i, a_{-i}) - u_j(a_i, a_{-i})] > 0$. Generically there are no indifferences, and hence G and G^c are better reply equivalent. Define a function $P : A \longrightarrow \mathbb{R}$ by $P(a) = \bar{u}_i(a)$, for all $a \in A$, and note that P is a potential for G^c. □

Theorems 1 and 2 have the following corollary.

Corollary 2. *Generically an ordinal potential game or a game with the FIP is better reply equivalent to a potential game.*

Hence from a strategic point of view, there is not much difference between potential games and games with FIP. We state the following counterpart to the previous for games with utilitarian best replies.

Theorem 3. *A game $G = \{N; (A_i)_i; (u_i)_i\}$ has the FBRP, iff G is best reply equivalent to a game $G' = \{N; (A_i)_i; (u'_i)_i\}$ with utilitarian best replies.*

Proof. Suppose that $G = \{N; (A_i)_i; (u_i)_i\}$ has utilitarian best replies. Take any $a^0 \in A$ and any best reply improvement path $(a^0, a^1, \ldots)$ starting from a^0. Then if a^0 is not already an equilibrium, we have that $\sum_i u_i(a^t) < \sum_i u_i(a^{t+1})$ as long as a^{t+1} is defined for $t \geq 0$. Since A is finite, the path $(a^0, a^1, \ldots)$ is finite and hence G has the FBRP.

Suppose then that G has the FBRP. Let A^0 denote the set of all equilibrium strategy profiles. A^0 is nonempty since G has the FBRP. Given $A^k, k \geq 0$, let A^{k+1} denote the subset of all those strategy profiles a from which A^k can be reached by a single BR-improvement. That is, $a \in A^{k+1}$ and $(b_i, a_{-i}) \in A^k$, iff $a_i \notin B_i(a), b_i \in B_i(A)$ for some i. Since A is finite, there is only finitely many subsets A^k and they from a partition of A.

Let $G' = \{N; (A_i)_i; (u'_i)_i\}$, where the utilities u'_i for all players are defined by $u'_i(a) = -k$ for $a \in A^k$. Denote the best replies of player i in the games G and G' by B_i and B'_i, respectively. Take any $a \in A$ such that $a_i \notin B_i(a)$. Then if $a \in A^{k+1}$ and $b_i \in B_i(a)$, we have that $(b_i, a_{-i}) \in A^k$ and $a_i \notin B'_i(a), b_i \in B'_i(a)$. On the other hand, if $a \in A^{k+1}$, $a_i \notin B'_i(a)$, and

$b_i \in B_i'(a)$, then $(b_i, a_{-i}) \in A^k$ and hence $a_i \notin B_(a), b_i \in B_i(a)$. Therefore G and G' are best reply equivalent.

If $a_i \notin B_i'(a)$ and $b_i \in B_i'(a)$, then $a \in A^{k+1}$ and $(b_i, a_{-i}) \in A^k$ for some k. This implies that $\sum_j u_j'(a) = nk < n(k+1) = \sum_j u_j'(b_i, a_{-i})$, and hence G' has utilitarian best replies. □

3.2. Weakly acyclic finite games

Let us now look at some ways of weakening the conditions of utilitarian preferences and best replies while maintaining the property that at least some improvement paths or BR-improvement paths converge. Weakly acyclic games have the property that starting from any non-equilibrium profile a^0, there is some improvement path that ends at an equilibrium. First we will introduce a version of a potential that fits to weakly acyclic games. Denote by $N(a)$ the subset of those players for whom $a_i \notin B_i(a)$.

Definition 2. A function $P : A \longrightarrow \mathbb{R}$ is a weak potential for a game $G = \{N; (A_i)_i; (u_i)_i\}$, if for any non-equilibrium profile $a \in A$, there exists $i \in N(a)$ and $b_i \in A_i$ such that $u_i(b_i, a_{-i}) > u_i(a)$ and $P(b_i, a_{-i}) > P(a)$. If b_i can be chosen to be a best reply at each $a \notin B(a)$, then P is called a weak best reply potential (weak BR-potential).

Kukushkin (2004) calls this a weak *numeric* potential. Note that since A is finite, a weak potential P has a maximum at some $a \in A$, and such a profile is an equilibrium. The converse need not hold: a could be an equilibrium and still not maximize P. Note also that every generalized ordinal potential is a weak potential but that the converse need not hold. The following result follows immediately from Definition 2, but we state for the sake of completeness.

Proposition 1. *A game $G = \{N; (A_i)_i; (u_i)_i\}$ is weakly acyclic, iff G has a weak potential. A game $G = \{N; (A_i)_i; (u_i)_i\}$ is weakly BR-acyclic, iff G has a weak BR-potential.*

Definition 3. A game $G = \{N; (A_i)_i; (u_i)_i\}$ has weakly utilitarian preferences, if $a \notin B(a)$ implies that $\sum_j u_j(a) < \sum_j u_j(b_i, a_{-i})$, for some $i \in N(a)$, for some $b_i \in A_i$ such that $u_i(a) < u_i(b_i, a_{-i})$. If b_i can be chosen to be a best reply at each $a \notin B(a)$, then we say that G has weakly utilitarian best replies.

Theorem 4. *If a game $G = \{N; (A_i)_i; (u_i)_i\}$ has weakly utilitarian preferences, then G is weakly acyclic. If $G = \{N; (A_i)_i; (u_i)_i\}$ has weakly utilitarian best replies, then G is weakly BR-acyclic.*

Proof. If a game $G = \{N; (A_i)_i; (u_i)_i\}$ has weakly utilitarian preferences, let $P(a) = \sum_j u_j(a)$, for each $a \in A$. Then P is a weak potential for G, and hence G is weakly acyclic.

Suppose a game $G = \{N; (A_i)_i; (u_i)_i\}$ has weakly utilitarian best replies. Let $(a^0, \ldots)$ be any BR-improvement path that satisfies

$$\sum_j u_j(a^t) < \sum_j u_j(a_i^{t+1}, a_{-i}^t), t \geq 0.$$

Then such a path cannot have a cycle, and since A is finite, there is some a^T where the path must end. Since G has weakly utilitarian best replies, a^T is an equilibrium. Hence G is weakly BR-acyclic. □

If $G = \{N; (A_i)_i; (u_i)_i\}$ has a weak potential P, and an improvement path $(a^0, \ldots)$ satisfies $P(a^t) < P(a^{t+1})$, then we say that the improvement path is *generated* by P.

Theorem 5. *If $G = \{N; (A_i)_i; (u_i)_i\}$ has a weak potential P, then there is a game $G' = \{N; (A_i)_i; (u'_i)_i\}$ with weakly utilitarian preferences such that every maximal improvement path of G that is generated by P is a maximal improvement path of G'. If G has a weak BR-potential P, then this holds for some G' that has weakly utilitarian best replies.*

Proof. If $G = \{N; (A_i)_i; (u_i)_i\}$ has a weak potential P, then let $G' = \{N; (A_i)_i; (u'_i)_i\}$ be such that $u'_i = P$. Then any maximal improvement path of G that is generated by P, is a maximal improvement path of G'.

If G has a weak BR-potential P, the the proof is like in the previous paragraph. □

3.3. Infinite games

Note that the definitions of various potentials (ordinal, generalized ordinal, weak) and utilitarian preferences and best replies do not depend on the cardinality of strategy sets or on any topological properties of these sets and utility functions. It follows that Theorem 1 and Proposition 1 for example hold also if strategy sets are infinite. Not all the previous results extend to infinite games so easily and we must specify more exactly the class of games.

Let $G = \{N; (A_i)_i; (u_i)_i\}$ be a game such that each A_i is a compact metric space. If all the utility functions are continuous on A, we call such a game *continuous*. Denote by $E(G)$ the set (possibly empty) of pure Nash equilibria of a game G.

Kukushkin (2011) studies acyclic games (he allows general compact strategy sets) that satisfy the following assumption. If $u_i(y_i, x_{-i}) > u_i(x)$, then there is a open neighborhood U of x_{-i} such that $u_i(y_i, z_{-i}) > u_i(x)$ for every $z_{-i} \in U$. That is, a strategy remains a better reply if opponents' strategies are perturbed only slightly. Utility functions u_i are assumed to be upper semicontinuous w.r.t. strategy profiles a and continuous w.r.t. opponents' strategies a_{-i}. Kukushkin (2011) shows that given initial profile a^0, there exists a Nash equilibrium $\bar{a}$ such that for any open neighborhood V of a^0, there exists a finite improvement path ending in V.

The following example is adapted from Example 1 in Salonen and Vartiainen (2010) (see also Example 1 in Kukushkin 2011, Remark 4.3.), and it shows that Kukushkin's result doesn't hold for BR-improvement paths.

Example 1. Let G be a two-person game. Let $A_i \subset \mathbb{R}^2$ be the boundary of the closed unit ball with center at the origin. Given a point $x \in A_j$, let $d(y, x)$ measure the distance of $y \in A_j$ from x along the boundary, with the qualification that if the distance is more than 1, then $d(y, x) = 1$. Define a utility function u_i on $A_1 \times A_2$ such that $u_i(a_i, a_j) = 1 - d(a_i, a_j + 1)$, where $a_j + 1 \in A_j$ is located clockwise right from a_j and has distance 1 from a_j. Utilities are continuous, best replies are unique ($B_i(a) = a_j + 1$), there are no best reply cycles but also no Nash equilibria.

Assuming that strategy sets are compact metric spaces Kukushkin's result holds also for BR-improvement paths.

Theorem 6. *If $G = \{N; (A_i)_i; (u_i)_i\}$ is acyclic, and utility functions u_i are upper semicontinuous w.r.t. profiles a and continuous w.r.t. opponents' strategies a_{-i}, then $E(G)$ is nonempty and closed. For any strategy profile a^0, for any $\varepsilon > 0$, there exists a BR-improvement path $(a^0, \ldots, a^T)$ such that a^T is within ε distance from $E(G)$.*

Proof. Choose $a^0 \in A$ such that a_0 is *not* an equilibrium. Let $\varepsilon > 0$ be such that $A_i \not\subset B(a_i^0, \varepsilon)$, where $B(a_i^0, \varepsilon)$ is the open ε -ball around a_i^0, for all $i \in N$. Given $\varepsilon/k, k \geq 1$, let $V_i(k)$ be the open ε/k covering of A_i. Since A_i is compact $V_i(k)$ has a finite subcovering $F_i(k)$. Let $A(k)_i$ be the set of the centers of the finitely many open balls in the union $F_i(0) \cup \cdots \cup F_i(k)$. Note that $A_i(k) \subset A_i(k+1)$ and that the closure of $\bigcup_k A_i(k)$ is A_i. Let G^k be the finite game with player set N, in which i has strategy set $A_i(k)$, and his utility function is u_i restricted to $A(k)$.

Let p^k be a maximal BR-improvement path of G^k starting from a^0. Then the last element $\bar{a}^k$ of the path p^k is an equilibrium of G^k. Since the original game G is acyclic, p^k is an improvement path of G.

The sequence $\{\bar{a}^k\}$ of last elements of paths p^k is a sequence in a compact metric space A, and hence it has a subsequence converging to some $\bar{a} \in A$. To save notation, assume w.l.o.g. that $\{\bar{a}^k\}$ itself converges. If $\bar{a}$ is not an equilibrium, then for some i and $b_i \in A_i$, we have $u_i(b_i, \bar{a}_{-i}) > u_i(\bar{a})$. But then for sufficiently large values of k, $u_i(b_i, a_{-i}^k) > u_i(\bar{a})$, since u_i is continuous w.r.t. a_{-i}. This shows that $E(G)$ is nonempty. The fact that $E(G)$ is closed follows from a similar argument.

For any $\varepsilon > 0$, we can choose $T = k$ large enough so that the distance between $\bar{a}^T$ and $\bar{a}$ is less than ε. □

If a continuous game G has a potential, then in fact the potential is continuous and its maximizers are Nash equilibria as observed by Monderer and Shapley (1996). However there are continuous games with an ordinal potential but without a continuous ordinal potential (Voorneveld 1997).

Theorem 1 says that in finite games there is a close relationship between ordinal potentials and utilitarian preferences. In continuous games, the same relationship holds between *continuous* ordinal potentials and utilitarian preferences.

Theorem 7. *A continuous game $G = \{N; (A_i)_i; (u_i)_i\}$ has a continuous ordinal potential, iff there is a continuous better reply equivalent game $G = \{N; (A_i)_i; (u_i')_i\}$ with utilitarian preferences.*

Proof. If $G = \{N; (A_i)_i; (u_i)_i\}$ has utilitarian preferences, then G has an ordinal potential $P = \sum_j u_j$ by Theorem 1, and P is continuous as a sum of continuous functions. If $G = \{N; (A_i)_i; (u_i)_i\}$ has a continuous ordinal P, then the game $G' = \{N; (A_i)_i; (u_i')_i\}$ is continuous and better reply equivalent to G when $u_i' = P$ for all i. □

For continuous games with a continuous ordinal potential we also get a rather strong approximation result for BR-improvement paths.

Theorem 8. *If G has a continuous ordinal potential, then from any $a^0 \in A$, there exists a BR-improvement path $(a^0, \ldots)$ such that for all $\varepsilon > 0$, all but finitely many elements of $(a^0, \ldots)$ are within ε distance from $E(G)$.*

Proof. If there is a finite improvement path starting from a^0 and ending at an equilibrium, we are done. If this does not hold, let $p = (a^0, a^1, \ldots)$ be a BR-improvement path, and note that such a path is necessarily infinite. We may choose this path in such a way that at each improvement $a^t \to a^{t+1}$ the strategy a_i^{t+1} maximizes $P(a_i^{t+1}, a_{-i}^t)$. This can be done since P is continuous and each A_i is compact.

Since P is an ordinal potential, $P(a^t) < P(a^{t+1}), t \geq 0$. Let $v = \sup\{P(a^t) \mid t \geq 0\}$. Since $\{a^t\}$ is a sequence in a compact metric space, it has a subsequence, say $\{a^{t_m}\}$, converging to some $\bar{a}$. By continuity of P, $v = P(\bar{a}) = \sup\{P(a^{t_m}) \mid t \geq 0\}$. If $\bar{a}$ is not an equilibrium, then for some i and some $b_i \in A_i$, we have that $u_i(b_i, \bar{a}_{-i}) > u_i(\bar{a})$. Since P is an ordinal potential, we have that $P(b_i, \bar{a}_{-i}) > P(\bar{a}) = v$.

By continuity of P, for sufficiently large value of m there must then exist an improvement $a^{t_m} \to (a'_j, a^{t_m}_{-j})$ (the deviating player j being i or somebody else) such that $P(a'_j, a^{t_m}_{-j}) > P(\bar{a}) = v$. But this is a contradiction, since at each stage a^t, the improvement $a^t \to a^{t+1}$ was chosen to maximize $P(a^{t+1})$.

Therefore $\bar{a}$ is a Nash equilibrium, and by the same method we can show that every cluster point c of the BR-improvement path $p = (a^0, a^1, \ldots)$ is a Nash equilibrium, and that $P(c) = v$. This completes the proof. □

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