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ABSTRACT

We study non-cooperative link formation games in which players have to decide how much to invest in relationships with other players. A link between two players is formed, if and only if both make a positive investment. The cost of forming a link can be interpreted as the opportunity cost of privacy. We analyze the existence of pure strategy equilibria and the resulting network structures with tractable specifications of utility functions. Sufficient conditions for the existence of reciprocal equilibria are given and the corresponding network structure is analyzed. Pareto optimal and strongly stable network structures are studied. It turns out that such networks are often complete.

JEL Classification: C72, D43

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1. Introduction

We study non-cooperative link formation games in which players have to decide how much to invest in relationships with other players. A link between two players is formed, if and only if both make a positive investment. The cost of forming a link can be interpreted as the opportunity cost of privacy. We analyze the existence of pure strategy equilibria and the resulting network structures with tractable specifications of utility functions. Sufficient conditions for the existence of reciprocal equilibria are given and the corresponding network structure is analyzed. Pareto optimal and strongly stable network structures are studied. It turns out that such networks are often complete.

Each player has a fixed amount of a single resource like time or effort that he can invest in relationships with other players and/or use for his own private benefit. The more two players invest in their mutual relationship, the higher is the utility to both players from this relationship. Since resources are limited, utility from privacy or from other relationships decreases, and there is a tradeoff between relationships. Decisions are made simultaneously and pure strategy Nash equilibria are searched for.

We show that a *reciprocal equilibrium* with a complete network exists in many symmetric or anonymous link formation games (Theorems 1 and 2). In such an equilibrium players i and j invest equal amounts in their mutual relationship. This amount may be different in relationships between different players.

Network structure in a reciprocal equilibrium depends on players' valuations of privacy. If these valuations are linear functions, then reciprocal equilibria often exhibit *homophily* (Theorem 3): links are more likely to be formed between similar players (Currarini *et.al* 2009).

Equilibria with a complete network exists under variety of circumstances when reciprocity is not demanded, for example in *semi-symmetric* games with bilateral strategic complements or substitutes (Theorems 4 and 5). In semi-symmetric link formation games players have common preferences over other players as friends.

In the class of models studied in this paper, Pareto optimality of a network structure implies in many cases that network must be complete (Propositon 1). Similarly, strongly stable equilibria (Bloch and Dutta 2009) have often complete networks as well (Proposition 2). Of course , completeness of a network sounds rather extreme if the player set is very large. A more moderate interpretation of these results would be that networks consist of a few cliques or perhaps of a few completely connected components.

Be this as it may, Bloch and Dutta (2009) get results that efficient or strongly stable networks are stars. It is therefore necessary to compare the underlying assumptions of our models.

We assume that players get utility only from private consumption or direct links (relationships) with other players, and that a relationship of two players gives positive utility only if both players have made a positive investment. Bloch and Dutta (2009) assume that players get utility also from indirect connections, *i.e.* from friends of friends, and that a link between two players is formed even if only one of the players has made a positive investment. In our model two linked players may value the relationship differently, whereas in their model the values are identical.

The model of Bloch and Dutta (2009) may be more natural in situations where links have instrumental value, like communication networks. Since direct links are not absolutely necessary for useful connections, complete networks need not be efficient structures. Our model is perhaps better suited in cases where links have intrinsic value, like friendships. In such cases indirect connections may be very poor substitutes for direct links, and increasing the number of direct links becomes both individually and socially optimal.

There is a large literature of link formation games where the link strength can take only two values: either it is 1 (link is formed) or 0 (link is not formed). Jackson and Wolinsky (1996) is the seminal paper of this strand of literature (see Jackson and Zenou 2014 for a comprehensive review of network games). Cabrales *et.al* (2011) analyze a linear quadratic game with productive investments and link formation where link strengths can be nonnegative real numbers. Rather than choosing each link intensity separately, a player chooses one real number that describes his socialization effort. Strengths of individual links are then determined jointly, given socialization efforts of all players. The resulting network determines the profitability of productive investments.

In our model players invest in each link separately, and the utility from equal investments in different links may be different. So the links of a player may represent very different relationships with other players, although seemingly a player decides only how to share a homogeneous resource among his friends.

The paper is organized in the following way. The notation is introduced in Section 2. In Section 3 some simple models with Cobb-Douglas functions are analyzed. Main results are stated in Section 4.

2. The Model

Given a *node set* N, the set of all (undirected, unweighted) *links* between elements of N is the set $g(N) = \{\{i,j\} \mid i,j \in N\}$.[1] Subsets $\{i,j\}$ we may be denoted by ij with the understanding that $ij = ji$. Given $g \subset g(N)$, a tuple $W = (N, g)$ is a *network* with a node set N and a link set g. If it is clear what the node set is we may denote a network simply by g. A network $W' = (N', g')$ is a *subnetwork* of $W = (N, g)$, if $N' \subset N$ and $g' \subset g$ such that $g' \subset g'(N')$. A network $W = (N, g)$ is complete if $g = g(N)$.

The set of *neighbors* of node i in a network g is $N_i(g) = \{j \neq i \mid ij \in g\}$. The *degree* of node i is the number $|N_i(g)|$ of his neighbors (other than i himself), denoted by $n_i(g)$. We adopt the convention that $i \notin N_i(g)$ although ii could be a link in g in our models.

[1]We give the main definitions concerning networks by using a simple undirected, unweighted network model. The networks actually studied in this paper are weighted, but the definitions for the simple networks extend naturally to the models analyzed here. All that is needed is to define link strengths $l(ij) > 0$ to links $ij \in g$. In directed networks $l(ij)$ may differ from $l(ji)$.

Given a network $W = (N, g)$ and $i, j \in N$, there exists a *path* P between i and j, if there exists nodes $i_0, \ldots, i_K$ such that i) $i_0 = i, i_K = j$; ii) $i_k i_{k+1} \in g$ for all $k = 0, \ldots, K-1$; iii) all nodes are distinct except possibly i_0 and i_K. A path P is a *cycle* if $i_0 = i_K$. A cycle is a *triad* if $K = 2$. A network $W = (N, g)$ is *connected* if there exists a path between any two nodes $i, j \in N$.

A subset $A \subset N$ is a *component* of a network $W = (N, g)$, if i) there exists a path between any two nodes $i, j \in A$; ii) there are no links between A and $A^c \equiv N \setminus A$. So a component A is a maximal connected subset of N. A *clique* is a subset $A \subset N$ such that all distinct nodes $i, j \in A$ are neighbors of each other. If a component is a clique, we may call it a complete component. If A is a clique then the subnetwork $W^A = (A, g^A)$ is a complete network, where g^A is the restriction of g on A.

A component A of a network $W = (N, g)$ is a *circle*, if A forms a cycle such that each $i \in A$ has exactly two neighhbors. A component A of a network $W = (N, g)$ is a *star*, if there is $i \in A$ such that i is the only neighbor of any $j \in A, j \neq i$. A network W is a circle or a star, if N is a circle or a star.

A *normal form game* $G = \big(N, (S_i)_{i \in N}, (u_i)_{i \in N}\big)$ specifies a player set N, a set of pure strategies S_i and a utility function $u_i : S \longrightarrow \mathbb{R}$ for each player $i \in N$, where $S = \Pi_i S_i$, the product of strategy sets, is the set of *strategy profiles*.

A game G is *symmetric*, if $S_i = S_j$ for all $i, j \in N$, and $u_i(s) = u_j(s')$ for all $i, j \in N$, for all $s, s' \in S$ such that $s_i = s'_j, s_j = s'_i$ and $s_k = s'_k$ for all $k \neq i, j$.

A game G is *anonymous*, $S_i = S_j$ for all $i, j \in N$, and $u_i(s) = u_i(s')$ if the only difference between s and s' is that $s_j = s'_k$ and $s_k = s'_j$ for some $j, k \neq i$.

Give $s \in S$, we may denote $s = (s_i, s_{-i})$ when we want to emphasize that i chooses s_i. A pure strategy Nash equilibrium is a strategy profile $s \in S$ such that

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}), \forall i \in N, \forall s'_i \in S_i. \tag{1}$$

Given a symmetric game G, a strategy profile s is a *symmetric equilibrium*, if $s_i = s_j$ for all $i, j \in N$.

We study *link formation games* of the following type. The set of pure strategies of player $i \in N$ is

$$S_i = \{s_i \in R_+^N \mid \sum_j s_{ij} = 1\}.$$

An interpretation is that each player i has one unit of time or effort to be shared with other player j including i himself. The utility function of player i is

$$u_i(s) = \sum_{j \neq i} U_{ij}(s_{ij}, s_{ji}) + V_i(s_{ii}), \tag{2}$$

where $U_{ij} : [0,1]^2 \longrightarrow \mathbb{R}_+$ is a function satisfying $U_{ij}(0, s_{ji}) = 0 = U_i(s_{ij}, 0)$ for all s_{ij}, s_{ji}, and $i, j \in N$. We assume that 1) U_{ij} is strictly concave and differentiable in s_{ij} for any given $s_{ji} > 0$; 2) U_{ij} is strictly increasing and continuous on $(0, 1] \times (0, 1]$.

Since investments s_{ij}, s_{ji} to the link between i and j could be different, the networks in this paper are actually directed and weighted. Players i and j get positive utility from a link if and only if they both make a strictly positive investment. This may be interpreted either so that there is no link between i and j unless both make an investment, or that there is a link if only one player makes an investment but that such a link gives no utility to either player in such a case.

The function $V_i : [0,1] \longrightarrow \mathbb{R}_+$ is concave, strictly increasing, differentiable on $(0, 1)$, and $V_i(0) = 0$. In anonymous link formation games $U_{ij} = U_i$ for all $i, j \in N$, and hence $u_i = U_i + V_i$. In symmetric link formation games $u_i = U + V$ for all $i \in N$. [In a link formation game the identity of strategies $s_i = s_j$ is understood so that $s_{ii} = s_{jj}$, $s_{ij} = s_{ji}$, and $s_{ik} = s_{jk}$ for all $k \neq i, j$.]

We say that a link formation game $G = \big(N, (S_i)_{i \in N}, (u_i)_{i \in N}\big)$ is *semi-symmetric*, if there are functions U and V such that

$$u_i(s) = \sum_{j \neq i} p_j U(s_{ij}, s_{ji}) + c_i V(s_{ii}), \forall s \in S, \tag{3}$$

for some parameters $p_j > 0, c_i > 0$, for all $i, j \in N$. So there is a common ordering of players such that player j is considered as a more valuable friend than i, if $p_j > p_i$. The cost parameters c_i reflecting the opportunity cost of privacy could be player specific.

Next we give some definitions that are needed in the main theorems.

Definition 1 (Bilateral strategic complements)**.** U_{ij} is twice continuously differentiable on $(0,1) \times (0,1)$ with $\partial^2 U_{ij}/\partial s_{ji}\partial s_{ij} > 0$, $i \neq j$.

Bilateral strategic complements imply $\partial^2 u_i/\partial s_{ji}\partial s_{ij} = \partial^2 U_{ij}/\partial s_{ji}\partial s_{ij}$ by equation 2. However, since $s_{ii} = 1 - \sum_{j \neq i} s_{ij}$ the usual strategic complements condition is not satisfied: if s_{ji} increases, a best reply may be such that that s_{ij} increases but s_{ik} decreases for some $k \neq j$.

Analogously, *bilateral strategic substitutes* means $\partial^2 U_{ij}/\partial s_{ji}\partial s_{ij} < 0$ holds on $(0,1) \times (0,1)$, for all players i.

Definition 2 (Increasing derivative on the diagonal)**.** A function U_{ij} : $[0,1]^2 \longrightarrow \mathbb{R}_+$ has (strictly) increasing derivative on the diagonal, if

$$\frac{\partial U_{ij}(y,y)}{\partial x_1} (<) \leq \frac{\partial U_{ij}(z,z)}{\partial x_1}, \text{ for all } y < z.$$

If the inequality is reversed, we say that U_{ij} has (strictly) decreasing derivative on the diagonal. If equality holds for all $y < z$ we say that U_{ij} has constant derivative on the diagonal.

Note that if U_{ij} is (jointly) concave, then it has a decreasing derivative on the diagonal. On the other hand the Cobb-Douglas function $f(x,y) = x^a y^b$ is concave in both arguments separately and has increasing derivative on the diagonal, if $0 < a, b < 1$, and $a + b \geq 1$. If U_{ij} is homogeneous of degree $\alpha \geq 1$ $(0 < \alpha \leq 0)$, then U_{ij} has increasing (decreasing) derivative on the diagonal. Homogeneity is clearly a much stronger assumption.

If a game is not symmetric, a symmetric equilibrium need not exist. However, behavior may be "nearly symmetric" also in non-symmetric games. The following is a "pairwise" or "bilateral" symmetry condition that seems natural in the context of friendship networks.

Definition 3 (Reciprocal equilibrium)**.** An equilibrium s of a link formation game $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$ is *reciprocal*, if $s_{ij} = s_{ji}$ for all players $i, j \in N, i \neq j$.

3. Examples

Let us first analyze some simple examples based on Cobb-Douglas functions U_{ij}.

Example 1. Let $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$ be a semi-symmetric game with bilateral strategic complements such that

$$u_i(s) = \sum_{j\neq i} p_j s_{ij}^{\alpha} s_{ji}^{1-\alpha} + c_i\big(1 - \sum_{j\neq i} s_{ij}\big),$$

where $0 < \alpha < 1$, $c_i, p_j > 0$. Given the value of α fixed, then for generic values of parameters c_i, p_j all equilibria s satisfying $s_{ii} > 0, \forall i$ are autarkic. That is, $s_{ii} = 1$. To see this, suppose that in equilibrium all the values of s_{12}, s_{21}, s_{11}, and s_{22} are strictly positive. Then the corresponding first order conditions for players 1 and 2 satisfy:

$$\alpha p_2 s_{12}^{\alpha-1} s_{21}^{1-\alpha} = c_1 \tag{4}$$

$$\alpha p_1 s_{21}^{\alpha-1} s_{12}^{1-\alpha} = c_2 \tag{5}$$

These equations imply

$$\alpha^2 p_1 p_2 = c_1 c_2,$$

which does not hold for generic values of c_i, p_j, given $\alpha \in (0, 1)$. If there are more than one player with the same parameters p_i, c_i, then equilibria typically exhibit *homophily*: links are formed only between similar players (Currarini *et.al* 2009).

On the other hand, an equilibrium need not be interior in order to have a complete network. As a simple example, consider a three-person game such that $\alpha = 1/2$, $p_1 = 1, p_2 = p_3 = 2$, and $c_i = 1$ for all i. One can verify that a profile s is an equilibrium, if $s_{12} = s_{13} = 1/2$, $s_{21} = s_{31} = 1/8$, and $s_{23} = s_{32} = 7/8$.

The game G has constant derivative on the diagonal and a linear V_i function. Theorem 2 below shows that if G is an anonymous game and V_i functions are strictly concave, then an interior *reciprocal* equilibrium often exists.

Let us modify the game of Example 1 slightly so that interior equilibria exist.

Example 2. Let $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$ be a game such that

$$u_i(s) = \sum_{j\neq i} p_j s_{ij}^{\alpha} s_{ji}^{\beta} + c_i\big(1 - \sum_{j\neq i} s_{ij}\big),$$

where $0 < \alpha, \beta$, $\alpha + \beta < 1$, and $c_i, p_j > 0$, for all $i, j \in N$. Let $p_{ij} = p_j/c_i$, and the first order conditions for an interior equilibrium for players i, j are:

$$\alpha p_{ij} s_{ij}^{\alpha-1} s_{ji}^{\beta} = 1 \tag{6}$$

$$\alpha p_{ji} s_{ji}^{\alpha-1} s_{ij}^{\beta} = 1 \tag{7}$$

Solving for s_{ij} gives us

$$s_{ij} = \alpha^{1/[1-\alpha-\beta]} \left[p_{ij}^{1-\alpha} p_{ji}^{\beta}\right]^{1/[(1-\alpha)^2-\beta^2]}, \forall i, j \in N. \tag{8}$$

If $p_{ij} = p$ for all i, j, then a symmetric interior equilibrium $s_{ij} > 0, s_{ii} > 0$ for all i, j exists if

$$\alpha p < \left[\frac{1}{n-1}\right]^{1-\alpha-\beta}.$$

For a nonsymmetric example, let $n = 11, \alpha = 1/4, \beta = 1/2$, and $p_1 = p, p_2 = p^2, \ldots, p_n = p^n$ for some $p \in (0, 1)$. If $c_i = 1$ for all i, then an equilibrium with a complete network is given by

$$s_{ij} = 4^{-4}\left[p^{2i+3j}\right]^{4/5}, \tag{9}$$

from which we can compute that

$$s_{ji} = \left[p^{i-j}\right]^{4/5} s_{ij}, \text{ for } j < i.$$

The players who are highly ranked by the society (high p_i) invest less in relationships than "low types".

For another numerical example, assume $p_j = 1$ for all players j, and $c_1 = c, c_2 = 2c, \dots, c_n = nc$, for some $c > 1/2$, and let the other parameters have the same values as above. Then the following values characterize an equilibrium with a complete network:

$$s_{ij} = 4^{-4}\left[i^{-3/4}j^{-1/2}c^{-5/4}\right]^{4/5}, \tag{10}$$

from which we can compute that

$$s_{ji} = \left[\frac{i}{j}\right]s_{ij}.$$

The players with high opportunity cost of privacy invest less in relationships than players with a low cost.

4. Results

Note first that the existence of a pure strategy equilibrium is not a problem in our model, since a strategy profile s such that $s_{ii} = 1$ and $s_{ik} = 0, k \neq i$, for all $i \in N$ is trivially an (autarkic) equilibrium, and also a reciprocal equilibrium. Here is a more interesting existence result for symmetric games.

Theorem 1. *A symmetric link formation game $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$ has a nontrivial reciprocal equilibrium with complete components, if and only if there exists $x \in (0,1)$ such that*

$$\frac{\partial U(x,x)}{\partial x_1} - V'(1-x) \geq 0. \tag{11}$$

Proof. ($\Leftarrow$) A reciprocal equilibrium s is nontrivial if $s_{ij} = s_{ji} > 0$ for at least two players i, j. Let $N_1, \dots, N_k$ be the complete components of the equilibrium network. If N_t has $m \geq 2$ members, then there exists $x = s_{ij}, i, j \in N_t$ such that

$$\frac{\partial U(x,x)}{\partial x_1} - V'(1-(m-1)x) \geq 0.$$

Since V is concave, the inequality 11 holds for this x.

($\Rightarrow$) Suppose that inequality 11 holds. Let m be the largest number, $m \leq n$, such that

$$\frac{\partial U(z,z)}{\partial x_1} - V'(1-(m-1)z) \geq 0$$

holds for some $z \in (0, 1/(m-1)]$. Clearly $m \geq 2$. Either there exists $z < 1/(m-1)$ such that this inequality is actually an equality, or else the inequality is satisfied by $z = 1/(m-1)$.

If $m = n$, then $s_{ij} = z$ for all $i, j \in N$, $j \neq i$, is a reciprocal equilibrium with a complete network. If $m < n$, then let k be the largest integer such that $km \leq n$. Choose k disjoint subsets N_t of N such that $|N_t| = m$ for all $t = 1, \ldots, k$. If the union of these subsets does not cover N, then let N_{k+1} be the residual subset.

Define $s \in S$ by setting $s_{ij} = z$ for all $i, j \in N_t$, $j \neq i$, $t = 1, \ldots, k$ (and also for $i, j \in N_{k+1}$ if this subset is nonempty) defines a nontrivial reciprocal equilibrium such that subsets N_t are complete components of the equilibrium network. □

Theorem 1 has the following corollary.

Corollary 1. *Suppose that a link formation game $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$ is such that equation 11 is satisfied. Then there exists a reciprocal equilibrium such that the corresponding network is a circle.*

Reciprocal equilibria may exist also in nonsymmetric games.

Theorem 2. *Let $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$ be an anonymous link formation game with the properties 1) constant derivative on the diagonal, 2) V_i is strictly concave. Assume also that if all players $i \in N$ have the same utility function u_i, then the corresponding symmetric game would have a symmetric equilibrium s^i such that the resulting network is complete, for any $i \in N$. Then there exists a reciprocal equilibrium such that the resulting network is complete.*

Proof. Since G is anonymous, $u_i = U_i + V_i$. By assumption, if all players have the same utility function u_i, there is a symmetric equilibrium s^i such that the resulting network is complete. Since u_i has constant derivative on the diagonal and V_i is strictly concave and increasing, the symmetric equilibrium s^i is unique. If every s^i is such that $s^i_{jk} = 1/(n-1)$, we are done. So we may assume that s^1 is the equilibrium in which $s^1_{ij} = x^1$ takes the smallest value, $i \neq j$, and $x^1 < 1/(n-1)$. Note that there may be another equilibrium s^k such that $s^k_{ij} = x^1$.

Construct a reciprocal equilibrium recursively as follows.

Step 1. Let N_1 be the subset of players for whom the following first order condition holds:

$$\frac{\partial U_i(x^1, x^1)}{\partial x} = V_i'\big(1 - (n-1)x^1\big). \tag{12}$$

By assumption, $|N_1| \geq 1$. If $N_1 = N$, the recursion ends. If $|N_1| < n$, then there exists at least one player for whom the left hand side of equation 12 is greater than the right hand side.

Step 2. Let $x^2 \in (0,1)$ be the least number such that $x^1 < x^2$ and the following weak inequality is satisfied for at least one player:

$$\frac{\partial U_i(x^2, x^2)}{\partial x_1} \geq V_i'\big(1 - n_1 x^1 - (n - n_1 - 1)x^2\big). \tag{13}$$

Since the derivative of U_i is constant on the diagonal and V_i is strictly concave, such an x^2 exists uniquely. Let N_2 be the set of players for whom equation 13 holds. If $|N_1| + |N_2| = n$, the recursion ends, because $N_1 \cap N_2 = \emptyset$. If $|N_1| + |N_2| < n$, continue the recursion to Step 3. Since there are n players, there is Step k, $k > 2$, as follows.

Step k. Let $x^k \in (0,1)$ be the least number such that $x^{k-1} < x^k$ and the following weak inequality is satisfied for at least one player:

$$\frac{\partial U_i(x^k, x^k)}{\partial x} \geq V_i'\Big(1 - \sum_{t<k} n_t x^t - (n_k - 1)x^k\Big). \tag{14}$$

By assumption and the previous Steps, such a number x^k exists uniquely. The subset N_k of players for whom equation 14 holds, satisfies $|N_1| + \cdots + |N_k| = n$ and $\{N_1, \ldots, N_k\}$ is a partition of N.

Given player $i \in N$, let m be such that $i \in N_m$. Define $s_{ij} = x^t$ for all $j \neq i$ such that $j \in N_t$ and $t < m$. For $j \neq i$ such that $j \in N_t$ and $m \leq t$, let $s_{ij} = x^m$. Let $s_{ii} = 1 - \sum_{t<m} n_t x^t - \big[\big(\sum_{m \leq t} n_t\big) - 1\big] x^m$.

By construction s is a reciprocal equilibrium such that the resulting network is complete. □

Note that if V_i is linear, then Theorem 2 may not hold by Example 1. Theorem fails if V_i is linear even if U_i is assumed to be strictly concave as the following result demonstrates.

Theorem 3. *Suppose $G = \big(N, (S_i)_{i \in N}, (u_i)_{i \in N}\big)$ is an anonymous link formation game such that 1) derivative is strictly decreasing on the diagonal; 2) V_i is linear and $U_i = U$. If there is a reciprocal equilibrium s such that the equilibrium network has a clique C and $s_{ii} > 0$ for all $i \in C$, then players $i \in C$ have the same utility functions $u_i = U + V_i$.*

Proof. By condition 2), $V_i(s_{ii}) = c_i s_{ii}$ for some constant $c_i > 0$. For each $i \in C$, there is at most one x^i such that $\partial U_i(x^i, x^i)/\partial x_1 = c_i$ by condition 1). For $i \in C$ this equality must hold in the reciprocal equilibrium s since $s_{ii} > 0$. If $c_i \neq c_j$, then $x^i \neq x^j$ because $U_i = U_j$. Therefore if C is a clique in an equilibrium network and $i, j \in C$, then $c_i = c_j$ and hence players in C have the same utility functions. □

Remark 1. Note that Theorem 3 holds also if condition 1) is replaced by the condition that derivative is strictly increasing on the diagonal. Of course, marginal utility from link formation may be so large as compared to the cost parameters c_i, that $s_{ii} = 0$ in equilibrium. Then there could exist reciprocal equilibria with a complete network even if players have different cost parameters c_i.

We show next that if a game has bilateral strategic complements, then with the same or slightly weaker assumptions as in Theorem 3 there exists an equilibrium such that the equilibrium network is complete. Of course, by Theorem 3 this equilibrium cannot be reciprocal.

Theorem 4. *Suppose* $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$ *is a semi-symmetric link formation game with bilateral strategic complements such that 1) derivative is strictly decreasing on the diagonal; 2) parameters* $p_j > 0$ *and* $c_i > 0$ *of Equation 3 are taken from compact intervals* P *and* C*, respectively; 3) the function* V *in Equation 3 is linear. Assume also that if all players* $i \in N$ *would have the same parameters* $p \in P, c \in C$*, then the corresponding symmetric game would have a symmetric equilibrium* s *such that the resulting network is complete and* $s_{ii} > 0$*, for all* $i \in N$*. Then there exists an equilibrium with a complete network.*

Proof. See Appendix. □

Remark 2. Note that Theorem 4 holds also if condition 1) is replaced by the condition that derivative is strictly increasing on the diagonal. In such a case an interior equilibrium is not stable in the usual best reply dynamics. The assumption of Theorem 4 that derivative is strictly decreasing (or strictly increasing) is critical as demonstrated in Example 1.

For games with bilateral strategic substitutes we have the following.

Theorem 5. *Suppose* $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$ *is a semi-symmetric link formation game with bilateral strategic substitutes such that 1) parameters* $p_j > 0$ *and* $c_i > 0$ *of Equation 3 are taken from compact intervals* P *and* C*, respectively; 2) the function* V *in Equation 3 is linear. If for each* p_j *and* c_i*, and for each* $z \in (0, 1/(n-1)]$ *there exists* $x \in (0, 1/(n-1)]$ *such that* $p_j \partial U(x, z)/\partial x_1 - c_i = 0$*, then there exists an equilibrium with a complete network.*

Proof. See Appendix. □

4.1. Efficiency and Stability of Equilibria

Given a game $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$, a strategy profile s is Pareto optimal, if there is no other profile s' such that $u_i(s') \geq u_i(s)$ for all $i \in N$ and $u_j(s') > u_j(s)$ for some $j \in N$. A network corresponding to a strategy profile s is Pareto optimal, if s is a Pareto optimal strategy profile. The following result gives some conditions under which a Pareto optimal network must be complete.

Proposition 1. *Suppose a link formation game* $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$ *is such that for each* $i \in N$ *and* $z^i \in (0,1]$ *there exists* $x^i \in (0, z^i)$ *such that*

$$\frac{\partial U_{ij}(x^i, x^i)}{\partial x_1} > V_i'(z^i - x^i), \forall i, j \in N, i \neq j,$$

and that each U_{ij} *is concave. If* $s \in S$ *is Pareto optimal and* $s_{ii} > 0, \forall i \in N$, *then* $s_{ij}, s_{ji} > 0$, $\forall i, j \in N$.

Proof. Suppose to the contrary that $s \in S$ is Pareto optimal and $s_{ii} > 0, \forall i \in N$, but $s_{ij} = 0$ for some $i, j \in N$. Since $U_{ij}(0, s_{ji}) = 0$ and $U_{ji}(s_{ji}, 0) = 0$, Pareto optimality of s implies that $s_{ji} = 0$. BY assumption, there exists $x^i < s_{ii}$ and $x^j < s_{jj}$ such that

$$\frac{\partial U_{ij}(x^i, x^i)}{\partial x_1} > V_i'(s_{ii} - x^i), \ \frac{\partial U_{ji}(x^j, x^j)}{\partial x_1} > V_j'(s_{jj} - x^j).$$

Since U_{ij} and V_i are concave functions, these inequalities hold for every $x \in (0, \min\{x^i, x^j\})$ as well. Given such an x, consider a strategy profile s' that is otherwise like the profile s, except that $s'_{ij} = s'_{ji} = x$, and $s'_{ii} = s_{ii} - x$, $s'_{jj} = s_{jj} - x$. Then $u_i(s') > u_i(s)$ and $u_j(s') > u_j(s)$ while $u_k(s') = u_k(s)$ for all $k \neq i, j$, and therefore s is not Pareto optimal, a contradiction. □

Remark 3. Proposition 1 holds for example when each U_{ij} is a strictly concave Cobb-Douglas function. The functions V_i can then be any concave, strictly increasing functions. Note that Proposition 1 holds also if functions U_{ij} have decreasing derivative on the diagonal, which is a weaker assumption than concavity.

An equilibrium s and the corresponding network are called s is *strongly pairwise stable*, if there is no strategy profile s' such that $u_i(s') > u_i(s)$ and $u_j(s') > u_j(s)$ for some $i, j \in N$, when $s_k = s'_k$ for all $k \in N \setminus \{i, j\}$ (Bloch and Dutta 2009). The following result states some conditions such that the network corresponding to a strongly stable equilibrium must be complete.

Proposition 2. *Suppose a link formation game* $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$ *is such that for each* $i \in N$ *and* $z^i \in (0,1]$ *there exists* $x^i \in (0, z^i)$ *such that*

$$\frac{\partial U_{ij}(x^i, x^i)}{\partial x_1} > V_i'(z^i - x^i), \forall i, j \in N, i \neq j,$$

and that each U_{ij} *is concave. If* $s \in S$ *is a strongly pairwise stable equilibrium and* $s_{ii} > 0, \forall i \in N$*, then* $s_{ij}, s_{ji} > 0$*,* $\forall i, j \in N, i \neq j$*.*

Proof. The proof of Proposition 1 applies here. □

Remark 4. If the functions V_i satisfy $\lim_{z\to 0^+} V_i'(z) = +\infty$, then $s_{ii} > 0$ must hold at any equilibrium.

Remark 5. The main lesson of Propositions 1 and 2 is *not* that Pareto optimal networks are complete, or that strongly stable equilibria have complete networks. Utilities from some links may be so low that these links are not formed either for efficiency or for equilibrium reasons. The lesson of these propositions is that network structures that have complete components often appear as efficient solutions or as equilibrium networks of a strongly pairwise stable equilibrium.

Usually in network literature efficiency is defined by using the utilitarian welfare function: those strategy profiles that maximize the sum of utilities are efficient. While such strategy profiles are Pareto optimal, not all Pareto optimal profiles satisfy this efficiency criterion.

If the functions U_{ij} are concave, then the utility functions u_i are concave on a simplex. In such a case each Pareto optimal strategy profile maximizes a *weighted* sum of players' utilities. The (positive) weights depend on the profile in question. If also the functions V_i satisfy $\lim_{z\to 0^+} V_i'(z) = +\infty$, then $s_{ii} > 0$ must hold at every Pareto optimal s, for all i.

5. Appendix

Proof of Theorem 4. Denote the set of "types" of players by $T = P \times C$. Given any type $t = (p, c) \in T$, if all players had this type, then by assumption there exists a symmetric interior equilibrium s^t satisfying

$$\frac{p}{c}\frac{\partial U(x^t,x^t)}{\partial x_1}=1$$

where $s^t_{ij}=x^t$ and $s^t_{ii}=1-(n-1)x^t$ for all $i,j\in N, j\neq i$. Since derivative is strictly decreasing on the diagonal, these symmetric equilibria can be ordered so that $x^t<x^{t'}$ iff $p/c>p'/c'$, where $t=(p,c)$ and $t'=(p',c')$.

Let $\bar{p}$ and $\underline{p}$ be the greatest and least elements, respectively, of the interval P. Define analogously $\bar{c}$ and $\underline{c}$. So the symmetric equilibrium corresponding to the type $\bar{t}=(\bar{p},\underline{c})$ has the largest x^t, denoted by $\bar{x}$. The symmetric equilibrium corresponding to the type $\underline{t}=(\underline{p},\bar{c})$ has the least x^t, denoted by $\underline{x}$.

Suppose that there are k different types $t^1,\dots,t^k\in T$ present in the player set N. Let N_m consists of all players whose type is $t^m, m=1,\dots,k$.

Let us construct an equilibrium s with a complete network such that players in the same subset N_m treat each other reciprocally.

Step 1. Set $s_{ii}=y^{t^m}$, and $s_{ij}=x^{t^m}$, for all $i,j\in N_m$, for all $m=1,\dots,k$. Note that the first order conditions of an interior equilibrium are satisfied by these choices. The values x^{t^m} and y^{t^m} are the same as in the symmetric equilibrium s^{t^m}.

Step 2. Take any players $i\in N_m$ and $j\in N_h$, $m\neq h$. Consider a two-person game with strategic complements between i and j. Let $t^m=(p,c)$ and $t^h=(p'c')$. Let b^t denote the best reply function of type $t=t^m,t^h$ against opponent's choices $x\in[\underline{x},\bar{x}]$. The best replies for these types (unique by strict concavity of $U(\cdot,x)$) satisfy

$$\frac{p'}{c}\frac{\partial U(b^{t^m}(x),x)}{\partial x_1}=1=\frac{p}{c'}\frac{\partial U(b^{t^h}(x),x)}{\partial x_1}.$$

If $p'/c=p/c'$, then best replies are the same. If $p'/c<p/c'$, then $b^{t^m}(x)<b^{t^h}(x)$. Since $\underline{p}/\bar{c}\leq p'/c<p/c'\leq\bar{p}/\underline{c}$, we have also $\underline{x}\leq b^t(\underline{x})\leq b^t(\bar{x})\leq\bar{x}$ for both types $t=t^m,t^h$. This holds since bilateral strategic complements implies $b^t(\underline{x})\leq b^t(\bar{x})$ (increasing best reply function). Strictly

decreasing derivative on the diagonal imply $\underline{x} \leq b^t(\underline{x})$ and $b^t(\bar{x}) \leq \bar{x}$, because $b^t(x^t) = x^t$. But then by Tarski's fixed point theorem the mapping $(x_m, x_h) \longrightarrow (b^{t^m}(x_h), b^{t^h}(x_m))$ on $[\underline{x}, \bar{x}] \times [\underline{x}, \bar{x}]$ has a fixed point (x^{mh}, x^{hm}).

Consider the game between all players in the set $N_m \cup N_h$. Then note that the choices y^{t^m}, x^{t^m}, x^{mh} for players in N_m and the choices y^{t^h}, x^{t^h}, x^{hm} for players in N_h form an equilibrium, since the resource constraints are satisfies by the definition of the symmetric equilibria $s^{t^1}, \ldots, s^{t^k}$, and the payoff of any player i is additively separable *w.r.t.* his opponents.

Since the types t^m and t^h were chosen arbitrarily, we have solved an equilibrium for the whole game. To see this, take any player i, and assume that $i \in N_m$. Then his choices satisfy the resource constraint:

$$y^{t^m} + |N_m - 1|x^{t^m} + \sum_{h \neq m} |N_h| x^{mh} = 1.$$

Since the first order conditions for maximum satisfied, we are done. □

Proof of Theorem 5. Let the "type set" be $T = P \times C$. Suppose that there are k different types $t^1, \ldots, t^k \in T$. Let N_m consist of all players whose type is $t^m, m = 1, \ldots, k$. We construct an equilibrium s such that players in the same subset N_{i_m} behave reciprocally.

Step 1. Suppose $i, j \in N_m$, so they both have the type $t^m = (p^m, c^m)$. Let $b^{t^m}(z)$ denote the unique best reply of either player to $z \in [0, 1/(n-1)]$. By assumption $b^{t^m}(1/(n-1)) \leq 1/(n-1)$. If equality holds, then $x^{t^m} = 1/(n-1)$ is a reciprocal equilibrium in the game with player set N_m.

Suppose $b^{t^m}(1/(n-1)) < 1/(n-1)$. Let $I^* = \{z \mid b^{t^m}(y) < y, \forall y \in [z, 1/(n-1)]\}$ and $x^* = \inf I^*$. Note that x^* exists since $1/(n-1) \in I^*$. We want to show that $b^{t^m}(x*) = x^*$.

By bilateral strategic substitutes, $b^{t^m}(1/(n-1) < b^{t^m}(z)$ for all $z \in I^*, z < 1/(n-1)$ and by assumption $b^{t^m}(1/(n-1)) > 0$. By the Theorem of the maximum, the best reply $b^{t^m}(z)$ is a continuous function on $[x^m - \varepsilon, 1/(n-1)]$, for any $\varepsilon > 0$ such that $x^* - \varepsilon > 0$. By continuity, $b^{t^m}(x^*) \leq x^*$. Again by continuity and the definition of I^*, this inequality cannot be strict, so $b^{t^m}(x^*) = x^*$. A reciprocal equilibrium in the game with player set N_m is obtained by setting $s_{ij} = x^* \equiv x^{t^m}$ for all $i, j \in N_m, i \neq j$.

Step 2. Suppose $i \in N_m$ and $j \in N_h$, $m \neq h$. Let $t^m = (p, c)$ and $t^h = (p', c')$. If $p'/c = p/c'$, then the choices given in *Step* 1 apply. Given $x \in [0, 1/(n-1)]$, the best replies satisfy

$$\frac{p'}{c}\frac{\partial U(b^{t^m}(x), x)}{\partial x_1} = 1 = \frac{p}{c'}\frac{\partial U(b^{t^h}(x), x)}{\partial x_1}.$$

Now $b^{t^m}(x) < b^{t^h}(x)$ because $U(\cdot, x)$ is strictly concave function, and because $p'/c < p/c'$.

Consider the function $f(x) = b^{t^h}(b^{t^m}(x))$ on $[b^{t^m}(1/(n-1)), 1/(n-1)]$. This function is continuous, and $f(x) \leq 1/(n-1)$ for all x. At $x = b_i(1/(n-1))$, $f(x) \geq x$, since both best replies are decreasing functions. Hence there is a fixed point $x^{hm} = f(x^{hm})$. But then x^{hm} is the best reply of player j against $b^{t^m}(x^{hm}) = x^{mh}$, which in turn is the best reply of player i against x^{hm}.

Therefore $s_{ji} = x^{hm}, s_{ij} = x^{mh}$ forms an equilibrium when the player set is $N_m \cup N_h$.

Since the types t^m and t^h were chosen arbitrarily, we have solved an equilibrium for the whole game. To see this, take any player i, and assume that $i \in N_m$. Then his choices satisfy the resource constraint:

$$|N_m - 1|x^{t^m} + \sum_{h \neq m} |N_h| x^{mh} \leq 1.$$

Define $s_{ii} = y^{t^m}$ so that the resource constraint is satisfied as equality. Since the first order conditions for maximum satisfied, we are done. □

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