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ABSTRACT

We consider optimal incremental capital accumulation in the presence of investment irreversibility and general price uncertainty. We present a set of general conditions under which the optimal capital accumulation path can be explicitly characterized in terms of an ordinary threshold rule stating that investment is optimal whenever the underlying price exceeds a capital-dependent threshold. We also present a set of general conditions under which increased price volatility expands the region where investment is suboptimal and decreases both the expected cumulative present value of the marginal revenue product of capital and the value of the future expansion options.

JEL Classification: G31, D92, C61

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1 Introduction

One of the basic conclusions of studies considering irreversible investment and sequential incremental capital accumulation in the presence of price uncertainty is that increased volatility increases the required exercise premium and expands the continuation region where investment is suboptimal (cf. Pindyck (1988), Dixit (1995), and Bertola (1998). For extensive and excellent surveys on the literature on irreversible investment, see Pindyck (1991) and Dixit and Pindyck 1994). The positivity of the sign of the relationship between increased volatility and the investment threshold typically follows from the observation that even though higher volatility may increase the expected cumulative present value of the marginal revenue product of the current capital stock it simultaneously increases the value of the opportunities to expand capacity later in the future. Since the latter effect dominates the former the existing literature concludes that increased price uncertainty should be detrimental for investment.

Even though the findings mentioned above are in line with economic intuition, they are obtained on models where the underlying driving processes evolve according to geometric Brownian motions. This naturally raises two important questions. First, given the relatively simple structure of the stochastic characterization of the underlying price dynamics it is not clear how a more general specification (for example, mean reverting) affects the optimal investment policy and its value. Second, given that the comparative static properties of the optimal policy and its value are highly sensitive with respect to the characterization of the underlying stochastically fluctuating unit price of output, it is not clear whether the negativity of the sign between increased volatility and the value remains true within a more general setting. Motivated by these arguments, we consider in this study the optimal incremental capital accumulation problem of a competitive firm in the presence of investment irreversibility and price uncertainty. In order to extend the results of previous studies addressing the same question we model the underlying stochastically fluctuating price as a one dimensional but otherwise general diffusion. Along the lines of previous studies modelling the price as a geometric Brownian motion, we find that the optimal capital accumulation policy can be characterized as an ordinary threshold policy stating that new productive capacity should be added as soon as the expected cumulative present value of the marginal revenue product of capital exceeds a critical level. We demonstrate that the optimal investment rule can be interpreted as a requirement that the firm should invest whenever the underlying price exceeds a capital-dependent threshold at which the expected cumulative net present value of the marginal revenue product of capital is maximized. Since each unit of stock decreases the marginal product of capital, we find that the optimal boundary is an increasing function of the capital stock. Therefore, in line with previous studies considering irreversible incremental capital accumulation our results indicate that small firms will generally invest more frequently than large ones in the general setting as well. We also consider how increased price volatility affects the optimal capital accumulation policy and its value and state in terms of the net appreciation rate of the unit price of output a set of general conditions under which higher price volatility unambiguously expands the continuation region where investment is suboptimal and decreases both the expected cumulative present value of the marginal revenue product of the current capital stock and the value of the future expansion options. Interestingly, since the conditions guaranteeing that the required exercise premium is an increasing function of volatility are not necessary for the existence of an optimal investment threshold, our results imply that the marginal value of capital does not generally have to be monotonic as a function of volatility. We also investigate the long run behavior of the optimal accumulation policy and state a set of conditions under which a well-defined long run stationary steady state distribution exists.

The contents of this paper are as follows. In section two we present the considered general model of capital accumulation and state our main results. These results are then explicitly illustrated in section three for two separate dynamic price specifications (geometric Brownian motion and a mean reverting diffusion) in the presence of a Cobb-Douglas production function. Section four then concludes our study.

2 Irreversible Capital Accumulation

Consider a competitive value maximizing firm facing a stochastically fluctuating price evolving on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ according to the stochastic dynamics characterized by the stochastic differential equation

$$dp_t = \mu(p_t)dt + \sigma(p_t)dW_t, \quad p_0 = p \in \mathbb{R}_+ \tag{2.1}$$

where W_t is standard Brownian motion, and both the drift coefficient $\mu : \mathbb{R}_+ \mapsto \mathbb{R}$ and the volatility coefficient $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ are assumed to be continuously differentiable. As usually, we denote as

$$\mathcal{A} = \frac{1}{2}\sigma^2(p)\frac{\partial^2}{\partial p^2} + \mu(p)\frac{\partial}{\partial p} \tag{2.2}$$

the differential operator associated with the price dynamics p_t. For simplicity, we will assume that the boundaries of the state space $(0, \infty)$ of the price process p_t are natural. Hence, even though the price dynamics may tend toward its boundary it is never expected to attain it in finite time (for a comprehensive characterization of the boundary behavior of linear diffusions, see Borodin and Salminen (2002), pp. 14–20).

The considered firm is assumed to produce a single homogenous output $F(k)$ by using a single homogenous productive input k, which is called capital. As usually, we assume that the function $F : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuously differentiable, monotonically increasing, strictly concave, and satisfies the *Inada*-conditions

$F(0) = 0$, $\lim_{k\downarrow 0} F'(k) = 0$ and $\lim_{k\to\infty} F'(k) = \infty$. Moreover, increasing the current operating capacity is assumed to be costly and that the unit cost of increased capacity is an exogenously determined constant $q > 0$. Given these assumptions, we now plan to analyze the optimal capital accumulation problem

$$V(k,p) = \sup_{k\in\Lambda} \mathbf{E} \int_0^\infty e^{-rs}(p_s F(k_s)ds - qdk_s), \tag{2.3}$$

where $r > 0$ is an exogenously given discount rate. Along the lines of Dixit (1995) we assume that the capital stock is non-depreciating and call an *irreversible capital accumulation policy k admissible* if it is non-negative, non-decreasing, right-continuous, and $\{\mathcal{F}_t\}$-adapted, and denote the set of admissible accumulation policies as Λ. Applying the generalized Itô-formula to the linear mapping $e^{-rt}qk$ yields (cf. Protter (1990), p. 74)

$$V(k,p) \leq qk + \sup_{k\in\Lambda} \mathbf{E} \int_0^\infty e^{-rs}(p_s F(k_s) - rqk_s)ds. \tag{2.4}$$

Thus, (2.4) demonstrates that the value of the firm is dominated by the sum of the value of current capital stock and the expected excess return accrued from following an optimal investment policy. It is worth emphasizing that since the inequality (2.4) becomes an equality whenever the expected present value of the future capital stock vanishes in the long run, (2.4) actually constitutes an explicit decomposition of the value in that case. In order to guarantee the finiteness of the objective functional (2.3) we assume that the expected cumulative present value of the maximal short run profit flow $\pi(p) = \sup_{k\in\mathbb{R}_+}[pF(k) - rqk]$ is bounded.

Instead of tackling the stochastic capital accumulation problem directly via variational inequalities (cf. Kobila (1993) and Øksendal (2000)), we rely on the optimal timing interpretation of the marginal value of capital (cf. Bertola (1998) and Pindyck (1988)) and re-express it as

$$V_k(k,p) = \inf_{\tau} \mathbf{E} \left[\int_0^\tau e^{-rs} p_s F'(k)ds + e^{-r\tau}q \right]. \tag{2.5}$$

The major advantage of this approach is that we can now focus on a single marginal investment decision instead of having to analyze the whole accumulation policy at once. Moreover, since the marginal value of capital can be interpreted as *Tobin's marginal q* focusing on the marginal decision is helpful in providing useful interpretations in terms of this classical capital theoretic approach (for an excellent and extensive survey of the q-theory of investment see Caballero (1999); for a critical treatment see Caballero and Leahy (1996)).

Applying now the strong Markov property of diffusions implies that (2.5) can be restated as

$$V_k(k,p) = G(p)F'(k) - \sup_{\tau} \mathbf{E}\left[e^{-r\tau}\left(G(p_\tau)F'(k) - q\right)\right], \tag{2.6}$$

where

$$G(p) = \mathbf{E}\int_0^\infty e^{-rs} p_s ds \tag{2.7}$$

denotes the expected cumulative present value of the flow p from the present up to an arbitrary distant future. It is well-known that if this value exists it can be re-expressed as

$$G(p) = B^{-1}\varphi(p)\int_0^p \psi(y) y m'(y) dy + B^{-1}\psi(p)\int_p^\infty \varphi(y) y m'(y) dy, \tag{2.8}$$

where B denotes the constant Wronskian of the fundamental solutions $\psi(p)$ and $\varphi(p)$ of the ordinary second order differential equation $(\mathcal{A}u)(p) = ru(p)$, $m'(p) = 2/(\sigma^2(p)S'(p))$ denotes the density of the speed measure and

$$S'(p) = \exp\left(-\int \frac{2\mu(p)dp}{\sigma^2(p)}\right)$$

denotes the density of the scale function of the price process p_t (for a complete characterization of the fundamental solutions, see Borodin and Salminen (2002), pp. 18–19). Moreover, since

$$G(p)F'(k) - q = \mathbf{E}\int_0^\infty e^{-rs}[p_s F'(k) - rq] ds$$

we observe that the optimal timing problem (2.6) states that investment is optimal at the instant when the difference between the expected cumulative present value of the future marginal revenue products of capital and the acquisition cost of a marginal unit of capacity is at its maximum. Since the marginal product of capital vanishes as the operating capital stock becomes unbounded, we find that the marginal value of future expansion options tends to zero as the operating capital stock becomes infinitely large.

Given these observations, we can now establish the following theorem characterizing the marginal value of the optimal capital accumulation policy and the exercise threshold at which investing is optimal.

Theorem 2.1. *Under the optimal capital accumulation policy the marginal value of capital reads as*

$$V_k(k,p) = \begin{cases} q & p \geq p^*(k) \\ \left[G(p) - \frac{G'(p^*(k))}{\psi'(p^*(k))}\psi(p)\right] F'(k) & p < p^*(k) \end{cases} \tag{2.9}$$

where the optimal investment threshold

$$p^*(k) = \underset{p}{\operatorname{argmax}} \left\{ \frac{G(p)F'(k) - q}{\psi(p)} \right\} > \frac{rq}{F'(k)} \tag{2.10}$$

is the unique root of the optimality condition $J(p^*(k))F'(k) = q$*, where*

$$J(p) = \frac{S'(p)}{\psi'(p)} \int_0^p \psi(y) y m'(y) dy \in [0, p/r] \tag{2.11}$$

is a continuously differentiable and monotonically increasing function. Especially, the marginal value of capital is a non-increasing function of the current capital stock and it satisfies the value matching condition $\lim_{p\to p^*(k)} V_k(k,p) = q$*, the smooth fit condition* $\lim_{p\to p^*(k)} V_{kp}(k,p) = 0$*, and the limiting conditions* $\lim_{k\to\infty} V_k(k,p) = 0$ *and* $\lim_{k\downarrow 0} V_k(k,p) = q$ *(for* $p > 0$*). Moreover, the optimal exercise threshold is an increasing function of the current operating capital stock and satisfies the conditions* $\lim_{k\to\infty} p^*(k) = \infty$*, and* $\lim_{k\downarrow 0} p^*(k) = 0$.

Proof. See Appendix A. □

Theorem 2.1 characterizes both the investment threshold at which investing is optimal and the marginal value of capital under the optimal capital accumulation policy. According to Theorem 2.1 the optimal capital accumulation policy can be characterized by a single investment threshold at which investment is optimal. Since the marginal revenue product of capital exceeds its user cost at the optimal boundary we find that uncertainty unambiguously increases the required exercise premium in comparison with the certainty case. Theorem 2.1 also establishes that the optimal investment boundary is an increasing function of the current operational capital stock. This observation is naturally based on the irreversibility of the capital accumulation policy and the monotonicity of the marginal product of capital. Since each marginal unit of installed capacity decreases the marginal revenue productivity of capital and, therefore, the value of future investment opportunities, a rationally investing firm has to increase the price at which investing becomes optimal in order to compensate for the lost option value. As usually in models considering single investment opportunities, we again find that the optimality condition characterizing the accumulation boundary has an intuitive interpretation in terms of the classical balance identity requiring that at the optimum the project value has to coincide with its full costs. More precisely, we find that at the optimum the expected cumulative present value of the revenue product generated by the acquired marginal unit of capacity has to coincide with the sum of its acquisition cost and the value of the lost expansion option. Theorem 2.1 also proves that the marginal value of capital is non-increasing as a function of the operating stock and, therefore, that the value function is concave as a function of capital. Thus, even though the marginal value of capital is positive it is non-increasing and eventually vanishes as the installed capital stock becomes infinitely large. However, the monotonicity of the optimal investment boundary implies that in terms of the current capital stock the investment rule states that for a fixed price p investment is optimal as long as the current capital stock satisfies is below the critical stock

$k^*(p) = p^{*-1}(p)$. Thus, for any positive current price p the value of a marginal unit of capacity coincides with its acquisition cost for capital stocks below the optimal boundary $k^*(p)$. A third important, and very natural, implication of Theorem 2.1 is that the value of the future expansion options vanish as the operating capacity tends to infinity. Thus, even though a greater capital stock increases the value of the firm it simultaneously decreases the incentives for further expanding the productive stock. Since the marginal product of capital vanishes as capacity tends to infinity the investment incentives vanish eventually as well.

In order to analyze how the optimal irreversible accumulation policy differs from the case where investment is reversible, we first notice that that the auxiliary functional stated in (2.11) can be re-expressed as $J(p) = (1 - \nu(p))p/r$, where[1]

$$\nu(p) = \int_0^1 \frac{Q(xp)}{Q(p)} dx \in (0, 1)$$

and $Q(p) = \psi'(p)/S'(p)$ is a positive, continuously differentiable, and monotonically increasing mapping. Thus, we find that along the optimal accumulation path investment occurs at the dates when the identity

$$pF'(k) = \left(1 + \frac{\nu(p)}{1 - \nu(p)}\right) rq \tag{2.12}$$

holds. It is worth emphasizing that if investment would be reversible, then the marginal revenue product of capital would coincide along the optimal capital accumulation path with its marginal user cost rq. Hence, (2.12) demonstrates that the term $\nu(p)/(1 - \nu(p))$ measures the required excess rate of return (from a marginal unit of stock) arising from the irreversibility of investment.

[1]Noticing that

$$J(p) = \frac{S'(p)}{\psi'(p)} \int_0^p \int_0^y \psi(y) m'(y) dx dy$$

and changing the order of integration (i.e. applying Fubini's theorem) yields

$$J(p) = \frac{p}{r} - \frac{1}{r} \frac{S'(p)}{\psi'(p)} \int_0^p \frac{\psi'(x)}{S'(x)} dx.$$

The optimal incremental capital accumulation policy and the resulting marginal product of capital are now explicitly characterized in the following.

Theorem 2.2. *The optimal capital accumulation policy reads as*

$$k_t^* = \max\left(k, {F'}^{-1}(q/J(\mathcal{M}_t))\right), \tag{2.13}$$

where $\mathcal{M}_t = \sup\{p_s; s \in [0,t]\}$ *denotes the maximum price up to time* t*. Hence, under the optimal capital accumulation policy, the marginal product of capital evolves according to the dynamics*

$$F'(k_t^*) = \min(F'(k), q/J(\mathcal{M}_t)). \tag{2.14}$$

Proof. See Appendix B. □

Theorem 2.2 characterizes the optimal capital accumulation policy. As usually, the optimal policy depends on the initial capacity. If the initial stock k is below the optimal level $k^*(p) = {p^*}^{-1}(p)$ then an immediate lump-sum investment $k^*(p)-k$ is made. After that a marginal unit of capacity is added whenever the underlying price increases to the investment boundary $p^*(k)$. If the initial stock is, however, above the optimal threshold $k^*(p)$ then marginal units of capacity are added whenever the underlying price increases to the investment boundary $p^*(k)$. The explicit characterization of the marginal product of capital implies that along the optimal accumulation path we have the inequality $J(\mathcal{M}_t)F'(k_t^*) \leq q$. Especially, since the maximum process $\mathcal{M}_t$ increases only at those times where it coincides with the underlying price, i.e. when the condition $\mathcal{M}_t = p_t$ is satisfied, and the mapping $J(p)$ is monotonic, we find that the condition $J(p_t)F'(k_t^*) < q$ holds for almost all dates and that the capital stock is increased only whenever the inequality becomes an equality. Thus, the optimal singular capital accumulation policy is such that the capital stock is maintained above the critical stock ${F'}^{-1}(q/J(p_t))$ at all times.

An important implication of Theorem 2.1 and Theorem 2.2 is now summarized in the following.

Corollary 2.3. *The value of the optimally investing firm is*

$$V(k,p) = \begin{cases} G(p)F(k) + \psi(p)\int_k^\infty \frac{G'(p^*(y))F'(y)}{\psi'(p^*(y))}dy & k > k^*(p) \\ q(k - k^*(p)) + V(k^*(p), p) & k \leq k^*(p). \end{cases} \tag{2.15}$$

Proof. See Appendix C. □

Corollary 2.3 presents an explicit characterization of the value of an optimally investing firm. Since a discrete lump sum investment $k^*(p) - k$ is made whenever the initial capital stock is below the optimal level $k^*(p)$, we find that in that case an optimally investing firm has to incur an immediate cost $q(k^*(p)-k)$ in order to acquire $V(k, k^*(p))$ capturing the value of future operation. If the initial stock is above the optimal level $k^*(p)$ then no initial investment is made and the firm initiates production with the existing operational stock. In that case the value is constituted by two factors. The first captures the expected cumulative present value of the revenue product of capital generated by the current stock. The second term, in turn, captures the value of future expansion opportunities (cf. Dixit and Pindyck (1994), p. 365).

Our main findings characterizing the comparative static properties of the optimal capital accumulation policy and its marginal value are now summarized in the following:

Theorem 2.4. *Assume that the net appreciation rate $\mu(p)-rp$ is non-increasing and concave. Then the marginal value of capital is non-decreasing and concave as a function of the current price p and increased volatility decreases its value. Moreover, higher volatility increases the investment threshold $p^*(k)$ at which investing is optimal and, therefore, expands the continuation region $\{(k,p) \in \mathbb{R}^2_+ : p < p^*(k)\}$ where investing is suboptimal.*

Proof. See Appendix D. □

Theorem 2.4 extends standard comparative static results (cf. Pindyck (1988, 1991), Dixit (1995), Dixit and Pindyck (1994), pp. 369–372, and Bertola (1998)) to the case where the underlying price may evolve according to a more general diffusion than just ordinary geometric Brownian motion. According to Theorem 2.4 increased volatility decreases the marginal value of capital and expands the continuation region where investing is suboptimal whenever the expected growth rate of the net present value of a unit of output is decreasing and concave. The reason for this observation is that under the assumptions of Theorem 2.4 increased volatility decreases both the expected cumulative present value of the marginal revenue product of the current capital stock and the value of the future expansion options. It is, however, worth noticing that since the existence of an optimal exercise boundary does not depend on the convexity properties of the drift coefficient $\mu(p)$, the marginal value of capital needs not to be a monotonic function of volatility.

Given the negativity of the sign of the relationship between increased volatility and the investment threshold one could be tempted to argue that the impact of higher volatility on the optimal capital accumulation path should be negative. This argument is, however, typically not true since even though increased volatility may expand the continuation region where investment should be postponed it simultaneously speeds up the maximum process $\mathcal{M}_t$ and, therefore, increases the probability of attaining high prices in fixed time intervals. Whichever of these two opposite effects dominates then determine the net impact of increased volatility on investment. We will illustrate this argument explicitly in the following section in an explicitly parametrized example.

Having considered the optimal capital accumulation policy and its marginal value, we now proceed in our analysis to the long-run behavior of the optimal accumulation policy. Our main findings on the long-run stationary behavior of the optimal capital stock are now summarized in the following.

Theorem 2.5. *Assume that* 0 *is an attracting boundary for the underlying price process, that is, assume that* $\lim_{p\downarrow 0} S(p) < \infty$. *Then, the optimal capital stock* *converges towards the long run stationary value*

$$k^*_\infty = \max\left(k, {F'}^{-1}(q/J(\hat{M}))\right) \tag{2.16}$$

and the marginal product of capital converges towards the long run stationary value

$$F'(k^*_\infty) = \min(F'(k), q/J(\hat{M})), \tag{2.17}$$

where the global maximum $\hat{M} = \sup\{p_s, s \in \mathbb{R}_+\}$ *is distributed on* $[p, \infty)$ *according to the distribution*

$$\mathbb{P}_p[\hat{M} \leq z] = \frac{S(z) - S(p)}{S(z) - S(0)}. \tag{2.18}$$

Proof. See Appendix E. □

Theorem 2.5 characterizes those circumstances under which the optimal capital accumulation path is stationary and has a non-trivial long-run stationary distribution. According to Theorem 2.5 such distribution exists only if the lower boundary 0 is attracting for the underlying price process. Otherwise, a long-run steady state distribution does no exist and the capital stock will eventually become almost surely infinite (cf. Karatzas and Shreve (1991), pp. 345–346). Theorem 2.5 also characterizes the long run stationary behavior of the optimal capital accumulation policy and the marginal product of capital *explicitly* instead of considering the long run stationary behavior of functionals of these random variables. Dixit and Pindyck (1994) (pp. 372–373) investigated the long run behavior of the marginal revenue product of capital and Bertola (1998) investigated the long run behavior of the ratio between the marginal profitability of capital and the acquisition cost of a unit of stock. Given that the underlying processes where modelled as geometric Brownian motions, they found that the long run stationary distribution of the considered functionals tend towards a truncated geometric distribution (the logarithms tend towards a truncated exponential distribution). In the present case, we find from Theorem 2.2 that the capital stock is controlled in such a way that the functional $(1-\nu(p_t))p_t F'(k_t^*)$ is maintained at almost all times below the capitalized unit cost rq of capital goods and, therefore, that under the optimal accumulation policy $(1-\nu(p_t))p_t F'(k_t^*)$ constitutes a one dimensional process evolving on $(0, rq]$ and reflected at rq. Unfortunately, deriving the long run stationary distribution of this process in the present case without further simplifications is extremely difficult, if possible at all.

3 Illustration

In this section we plan to illustrate our main results in two explicit examples. In the first example we reconsider the standard case where the underlying price process is assumed to evolve according to a geometric Brownian motion. In the second example, we assume that the underlying price evolves according to a mean reverting process. In both cases, we assume that the production function is of the standard Cobb-Douglas form $F(k) = k^\beta$, where $\beta \in (0, 1)$ is a known exogenously given constant measuring the elasticity of production. In light of our findings, the optimal investment boundary now reads as

$$k^*(p) = \left(\frac{\beta J(p)}{q}\right)^{1/(1-\beta)},$$

where $J(p)$ is defined as in (2.11).

3.1 Exponentially Growing Prices

In this subsection we assume that the underlying price process evolves according to a geometric Brownian motion characterized by the stochastic differential equation

$$dp_t = \mu p_t dt + \sigma p_t dW_t, \quad p_0 = p \in \mathbb{R}_+, \tag{3.1}$$

where $\mu, \sigma \in \mathbb{R}_+$ are exogenously determined constants. It is known that in this case the increasing fundamental solution of the ordinary differential equation

$(\mathcal{A}u)(p) = ru(p)$ reads as $\psi(p) = p^{\eta}$, where

$$\eta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{3.2}$$

denotes the positive root of the characteristic equation $\sigma^2 a(a-1)/2+\mu a-r = 0$. Since $S'(p) = p^{-2\mu/\sigma^2}$ we observe that if the condition $r > \mu$ is satisfied then the optimal exercise boundary is (cf. Dixit and Pindyck (1994), p. 372)

$$p^*(k) = \frac{\eta}{\eta-1}\frac{(r-\mu)q}{\beta}k^{1-\beta} = \left(1-\frac{1}{\theta}\right)\tilde{p}(k),$$

where

$$\theta = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{3.3}$$

denotes the negative root of the characteristic equation $\sigma^2 a(a-1)/2 + \mu a - r = 0$ and $\tilde{p}(k) = rqk^{1-\beta}/\beta$ denotes the optimal investment threshold in the absence of uncertainty and irreversibility. Since $\partial\theta/\partial\sigma = 2\theta(1-\theta)/(\sigma(\theta-\eta)) > 0$ we observe that $\partial p^*(k)/\partial\sigma > 0$ and, therefore, that increased volatility unambiguously expands the continuation region where investing is suboptimal by increasing the optimal investment boundary.

Given the explicit characterization of the investment boundary we now find that the optimal capital accumulation policy reads as

$$k_t^* = \max\left(k, \left(\frac{(\eta-1)\beta}{\eta q(r-\mu)}\right)^{1/(1-\beta)} \mathcal{M}_t^{1/(1-\beta)}\right),$$

where $\mathcal{M}_t = \sup\{p_s; s \leq t\}$ denotes the maximum price up to the date t. If $p \geq p^*(k)$, then the marginal revenue product of the optimal capital stock at any future date t reads as

$$\beta p_t {k_t^*}^{\beta-1} = \frac{\eta q(r-\mu)}{(\eta-1)}\frac{p_t}{\mathcal{M}_t} = \frac{\eta q(r-\mu)}{(\eta-1)}e^{\sigma(\sup\{W_s;s\leq t\}-W_t)}.$$

Applying *Levy's* characterization of reflected Brownian motion (cf. Borodin and Salminen (2002), p. 55) implies that

$$\beta p_t {k_t^*}^{\beta-1} \sim \frac{\eta q(r-\mu)}{(\eta-1)}e^{-\sigma|W_t|}$$

and, therefore, that the expected marginal revenue product of the optimal capital stock is

$$\mathbf{E}_p\left[\beta p_t k_t^{*\beta-1}\right] = 2e^{\frac{1}{2}\sigma^2 t}\Phi\left(-\sigma\sqrt{t}\right)\frac{\eta q(r-\mu)}{(\eta-1)},$$

where $\Phi(x)$ denotes the cumulative distribution function of the standard normal distribution. The expected marginal revenue product of the optimal capital stock is illustrated for various volatilities and drift coefficients in Figure 1 under the parameter specifications $r = 0.03$, and $q = 10$. Interestingly, Figure

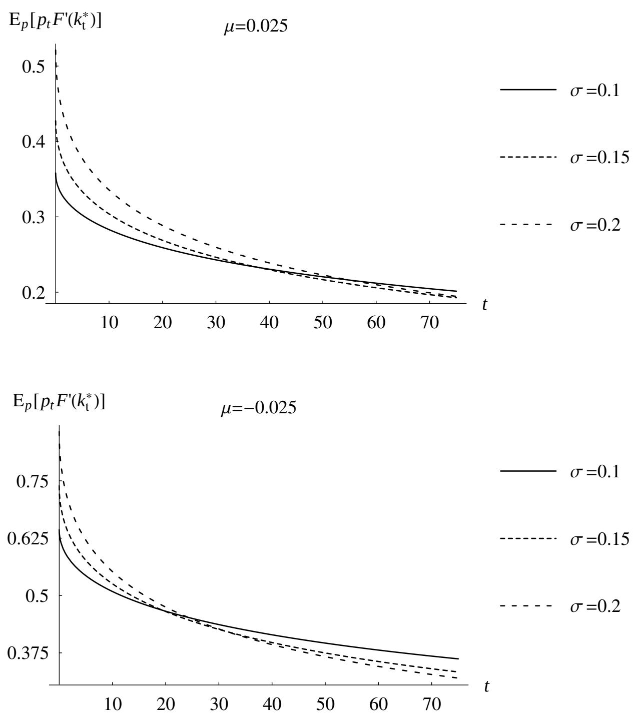

Figure 1: The Expected Marginal Revenue Product of Capital

1 indicates that even though increased volatility unambiguously increases the optimal investment boundary, its effect on the expected marginal revenue product of capital is ambiguous. In the specific circumstances of Figure 1, increased

price volatility has a positive impact on the marginal revenue product of capital in the short run but a negative one in the long run. Moreover, the expected capital stock now reads as (when $p \geq p^*(k)$)

$$\mathbf{E}_p\left[k_t^*\right] = 2e^{\frac{1}{2}\left(\frac{\sigma}{1-\beta}\right)^2 t}\Phi\left(\frac{\sigma\sqrt{t}}{1-\beta}\right)\left(\frac{\theta\beta p e^{(\mu-\sigma^2/2)t}}{(\theta-1)rq}\right)^{\frac{1}{1-\beta}}.$$

We illustrate the optimal capital stock for various volatilities and drift coefficients in Figure 2 under the parameter specifications $r = 0.03$, and $q = 10$. Along the lines of our findings on the marginal revenue product of the optimal

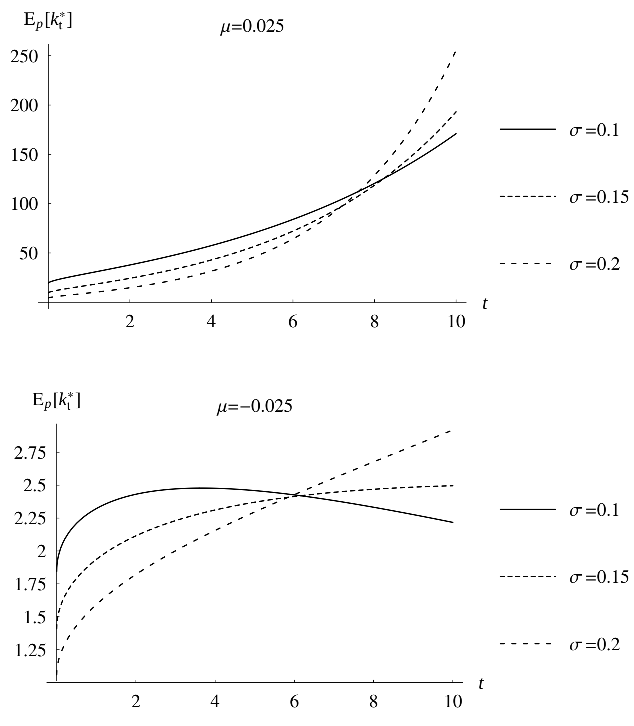

Figure 2: The Expected Optimal Capital Stock

capital stock, we again observe that the impact of increased volatility on the optimal capacity is ambiguous. Again, we observe that even though the impact of higher volatility on the capital stock may be negative in the short run, it appears to increase the stock in the long run.

It remains to consider the long-run behavior of the stock. If $\mu \geq \frac{1}{2}\sigma^2$ then $\mathcal{M}_t \to \infty$ almost surely as $t \to \infty$ and, therefore, in that case the optimal capital stock becomes almost surely unbounded in the long run. However, if $\mu < \frac{1}{2}\sigma^2$ then $\mathcal{M}_t \to \hat{M}$ where the global maximum $\hat{M}$ is distributed on $[p, \infty)$ according to the geometric distribution

$$\mathbb{P}_p[\hat{M} \leq z] = 1 - \left(\frac{p}{z}\right)^{\eta+\theta}.$$

In this case, if the integrability condition $\mu < -\frac{\beta\sigma^2}{2(1-\beta)}$ is satisfied and $p > p^*(k)$ then the expected long run capital stock is

$$\mathbf{E}_p[k^*_\infty] = \frac{(1-\beta)(\theta+\eta)}{(1-\beta)(\theta+\eta)-1}\left(\frac{\theta\beta p}{(\theta-1)rq}\right)^{1/(1-\beta)}.$$

The expected long run capital stock is illustrated for various volatilities in Figure 3 under the parameter specifications $\mu = -0.025, r = 0.03, q = 10$, and $\beta = 0.75$. As Figure 3 clearly indicates, whenever an expected long run capital stock exists, it is an increasing function of the volatility of the underlying unit price of output.

3.2 Mean Reverting Prices

In order to illustrate our general results in a dynamically more complex setting, we now assume that the underlying price process evolves according to a mean reverting diffusion characterized by the stochastic differential equation

$$dp_t = \mu p_t(1 - \gamma p_t)dt + \sigma p_t dW_t, \quad p_0 = p \in \mathbb{R}_+, \tag{3.4}$$

where $\mu, \gamma, \sigma \in \mathbb{R}_+$ are exogenously determined constants. It is known that in this case the increasing fundamental solution of the ordinary differential equation $(\mathcal{A}u)(p) = ru(p)$ reads as

$$\psi(p) = p^\eta M\left(\eta, 2\eta + \frac{2\mu}{\sigma^2}, \frac{2\mu\gamma}{\sigma^2}p\right),$$

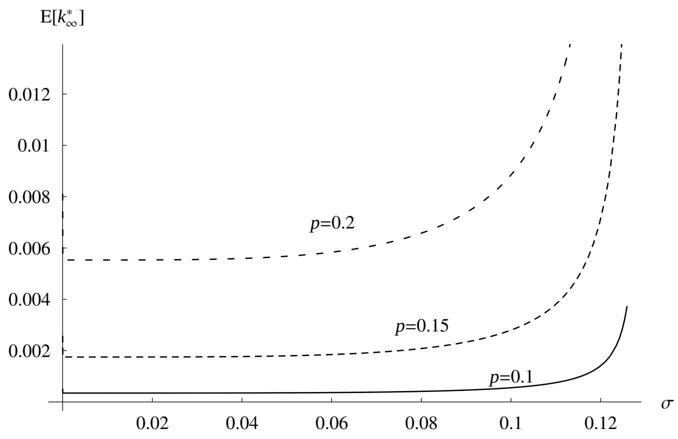

Figure 3: The Expected Long Run Capital Stock

where

$$\eta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

denotes the positive root of the characteristic equation $\sigma^2 a(a-1)/2+\mu a-r=0$, and M denotes the confluent hypergeometric function of the first type (cf. Abramowitz and Stegun (1968), pp. 555-566). Unfortunately, deriving the optimal boundary and the optimal capital accumulation policy explicitly is in this case impossible and, therefore, we illustrate the optimal investment boundary in Figure 4 under the parameter specifications $\mu = 0.02, r = 0.035, \gamma = 0.01, \beta = 0.5$, and $q = 10$. In accordance with the findings of Theorem 2.1, Figure 4 shows that the optimal exercise boundary is monotonically increasing as a function of the current capital stock. Moreover, as was established in Theorem 2.4 we find that increased price volatility increases the optimal investment boundary and, therefore, expands the continuation region where investing is suboptimal.

In this case the density of the scale function $S(p)$ reads as

$$S'(p) = p^{-2\mu/\sigma^2} e^{2\mu\gamma p/\sigma^2}.$$

Thus, if the condition $\mu \leq \frac{1}{2}\sigma^2$ is satisfied, then the global maximum $\hat{M} =$

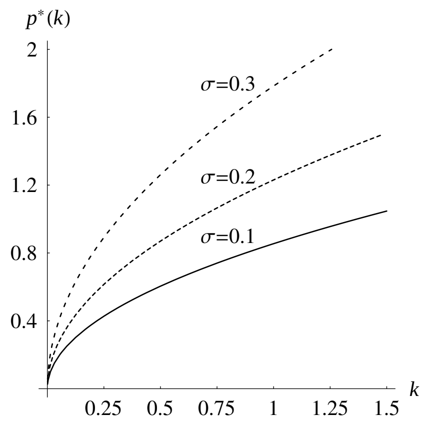

Figure 4: The Optimal Exercise Boundary Under Decreasing Returns to Scale

$\sup\{p_s; s \in \mathbb{R}_+\}$ is distributed on $[p, \infty)$ according to the truncated Gamma-distribution

$$\mathbb{P}_p[\hat{M} \leq z] = 1 - \frac{\int_0^{2\mu\gamma p/\sigma^2} y^{-2\mu/\sigma^2} e^y dy}{\int_0^{2\mu\gamma z/\sigma^2} y^{-2\mu/\sigma^2} e^y dy}.$$

Since this probability distribution is not monotonic as a function of volatility, we conjecture that the impact of increased volatility on the expected long run capital stock is ambiguous.

4 Concluding Comments

In this study we analyzed how price uncertainty and investment irreversibility affects the optimal capital accumulation policy of a competitive firm. We extended previous results to a general setting and established that the optimal capital accumulation policy is generally characterizable as a rule stating that a further marginal unit of capacity should be acquired whenever the marginal revenue product of capital exceeds an optimal threshold at which the expected cumulative net present value of the marginal revenue product of capital is maximized. We also analyzed the sign of the relationship between increased volatility and the rational capital accumulation policy and stated a set of general conditions under which increased volatility unambiguously expands the continuation region where investment is suboptimal and decreases the marginal value of capital.

There are several economically interesting directions towards which the analysis of our study could be extended. Given the perpetuity of the considered optimal investment problems a natural extension of our analysis would be to introduce interest rate uncertainty and in that way consider the impact of a stochastically fluctuating opportunity cost of investment on the optimal capital accumulation policy. Especially, such a generalization would indicate the main differences between the optimal investment policy in the sequential case where current capacity affects the future investment options and in the single investment opportunity case where the discrete investment opportunity is either exercised or not (cf. Ingersoll and Ross (1991) and Alvarez and Koskela (2003, 2005, 2006)). A second interesting extension would be to admit partial reversibility of investment by assuming that disinvestment is costly along the lines of the models considered in Abel and Eberly (1996) and Abel, Dixit, Eberly and Pindyck (1996). Such an extension would provide potentially valuable information on the impact of asymmetric investment costs on optimal capital accumulation policies in a general setting. A third natural extension of our analysis would be to model all factor prices, productivity growth, and the underlying interest rate dynamics as potentially dependent stochastic processes (cf. Bertola (1998)). Unfortunately, such extensions are out of the scope of the present study and left for the future.

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A Proof of Theorem 2.1

Proof. Consider first the mapping $J(p)$. Applying the identity

$$\frac{\psi'(p)}{S'(p)} = r \int_0^p \psi(y) m'(y) dy \tag{A.1}$$

implies that

$$J(p) = \frac{\int_0^p \psi(y) y m'(y) dy}{r \int_0^p \psi(y) m'(y) dy} < \frac{p}{r}$$

and, therefore, that $\lim_{p \downarrow 0} J(p) = 0$. Since $\lim_{p \to \infty} \psi'(p)/S'(p) = \infty$ we find by applying L'Hospitals rule that $\lim_{p \to \infty} J(p) = \lim_{p \to \infty} p/r = \infty$. Standard differentiation of $J(p)$ yields

$$J'(p) = \frac{r\psi(p)m'(p)}{(\psi'(p)/S'(p))^2} \int_0^p \psi(y)(p-y)m'(y)dy > 0$$

demonstrating that $J(p)$ is strictly increasing. Given these findings consider for a fixed initial capital stock $k \in \mathbb{R}_+$ the behavior of the mapping

$$L(k,p) = \frac{G(p)F'(k) - q}{\psi(p)}.$$

Differentiating $L(k,p)$ and applying the representation (2.8) yields

$$L_p(k,p) = \frac{\psi'(p)}{\psi^2(p)} \left[q - J(p)F'(k) \right].$$

Since $\lim_{p \to \infty} J(p) = \infty$ and $\lim_{p \downarrow 0} J(p) = 0$ the monotonicity of the mapping $J(p)$ implies that equation $L_p(k,p) = 0$ has a unique root $p^*(k) = \operatorname{argmax}_p \{L(k,p)\} > rq/F'(k)$. Implicit differentiation now yields

$$\frac{\partial p^*(k)}{\partial F'(k)} = -\frac{-J(p^*(k))}{J'(p^*(k))F'(k)} < 0$$

proving that increased productivity accelerates investment by decreasing the optimal exercise boundary. The strict concavity of $F(k)$ then implies that ${p^*}'(k) > 0$. Since the optimality condition can be re-expressed as $J(p^*(k)) = q/F'(k)$ we find by the existence and uniqueness of $p^*(k)$, the continuity of $J(p)$, and the assumption $\lim_{k \to \infty} F'(k) = 0$ that $\lim_{k \to \infty} p^*(k) = \infty$. Considering the case $\lim_{k \downarrow 0} p^*(k)$ is analogous.

Denote the proposed marginal value function as $U(k,p)$. Given the observations above we immediately observe that since the proposed marginal value function can be re-expressed as

$$U(k,p) = q + (G(p)F'(k) - q) - \psi(p) \sup_{y \geq p} \left[\frac{G(y)F'(k) - q}{\psi(y)} \right]$$

we have that $U(k,p) \leq q$ for all $(k,p) \in \mathbb{R}^2_+$. On the other hand, since $(\mathcal{A}U)(k,p) - rU(k,p) + pF'(k) = 0$ for all $p < p^*(k)$ and $(\mathcal{A}U)(k,p) - rU(k,p) + pF'(k) = pF'(k) - rq > 0$ for all $p > p^*(k)$, we find that $(\mathcal{A}U)(k,p) - rU(k,p) + pF'(k) \geq 0$ for all $(k,p) \in \mathbb{R}_+ \times (\mathbb{R}_+ \backslash \{p^*(k)\})$. Since $U(k,p)$ is continuous with respect to the current capital stock, twice continuously differentiable with respect to the current price outside the optimal boundary, $U_{pp}(k, p^*(k)+) = 0$ and

$$U_{pp}(k, p^*(k)-) = -\frac{2(p^*(k)F'(k) - rq)}{\sigma^2(p^*(k))} < \infty$$

we observe that the proposed marginal value function satisfies the sufficient variational inequalities (cf. Theorem 10.4.1 in Øksendal (2003), p. 225) and, therefore, that $U(k,p) \leq V_k(k,p)$ for all $(k,p) \in \mathbb{R}^2_+$. However, since

$$U(k,p) = \mathbf{E}\left[\int_0^{\tau(p^*(k))} e^{-rs} p_s F'(k) ds + e^{-r\tau(p^*(k))} q \right],$$

where $\tau(p^*(k)) = \inf\{t \geq 0 : p_t \geq p^*(k)\}$ we find that $U(k,p) \geq V_k(k,p)$ which demonstrates that $U(k,p) = V_k(k,p)$.

Under the optimal capital accumulation policy the marginal value of capital can be written on the continuation region $(0, p^*(k))$ where investing is suboptimal as

$$V_k(k,p) = G(p)F'(k) - \frac{G(p^*(k))F'(k) - q}{\psi(p^*(k))} \psi(p)$$

Standard differentiation then implies that $V_{kk}(k,p) = G(p)F''(k) < 0$. On the other hand, since $V_k(k,p) = q$ on the region where investing is optimal, we find that $V_{kk}(k,p) \leq 0$ on the investment region as well and, therefore, that the marginal value of capital is decreasing as a function of capital. It remains to establish that the marginal value of capital vanishes as the capital stock becomes infinitely large. To see that this is indeed the case, we first observe that since

$$\begin{aligned}\frac{G'(p^*(k))F'(k)}{\psi'(p^*(k))} &= B^{-1}\frac{\varphi'(p^*(k))}{\psi'(p^*(k))}\int_0^{p^*(k)}\psi(y)yF'(k)m'(y)dy \\ &\quad +B^{-1}\int_{p^*(k)}^{\infty}\varphi(y)yF'(k)m'(y)dy \\ &= B^{-1}\frac{\varphi'(p^*(k))}{S'(p^*(k))}q + B^{-1}\int_{p^*(k)}^{\infty}\varphi(y)yF'(k)m'(y)dy \\ &= B^{-1}\int_{p^*(k)}^{\infty}\varphi(y)(yF'(k)-rq)m'(y)dy\end{aligned}$$

the marginal value can be re-expressed on the continuation region $(0, p^*(k))$ where investing is suboptimal as

$$V_k(k,p) = G(p)F'(k) - \psi(p)B^{-1}\int_{p^*(k)}^{\infty}\varphi(y)(yF'(k)-rq)m'(y)dy.$$

Since $\lim_{k\to\infty} p^*(k) = \infty$ and $\lim_{k\to\infty} F'(k) = 0$ we finally find that $\lim_{k\to\infty} V_k(k,p) = 0$ completing the proof of our theorem. □

B Proof of Theorem 2.2

Proof. The separability of the optimality condition $J(p^*(k))F'(k) = q$ and the monotonicity of the marginal product $F'(k)$ implies that the optimal investment boundary can be expressed as $k = {F'}^{-1}(q/J(p))$. Hence, as was established in Øksendal (2000) (see also Kobila (1993)) the optimal capital accumulation policy constituting the solution of the associated Skorokhod-problem reads as

$$k_t^* = \max\left(k, \sup\{{F'}^{-1}(q/J(p_s)); s\in[0,t]\}\right).$$

The monotonicity of the marginal product of capital implies that

$$k_t^* = \max\left(k, {F'}^{-1}(\inf\{q/J(p_s); s\in[0,t]\})\right).$$

The monotonicity of the function $1/J(p)$ then implies that

$$k_t^* = \max\left(k, {F'}^{-1}(q/J(\sup\{p_s; s\in[0,t]\}))\right)$$

proving that the optimal policy reads as in (2.13). The monotonicity of the marginal product of capital then implies (2.14). □

C Proof of Corollary 2.3

Proof. Since

$$\frac{G'(p^*(k))F'(k)}{\psi'(p^*(k))} = B^{-1}\int_{p^*(k)}^{\infty}\varphi(y)(yF'(k)-rq)m'(y)dy$$

we find that

$$\int_k^{\infty}\frac{G'(p^*(v))F'(v)}{\psi'(p^*(v))}dv = B^{-1}\int_k^{\infty}\int_{p^*(v)}^{\infty}\varphi(y)(yF'(v)-rq)m'(y)dydv.$$

Changing the order of integration yields

$$\begin{aligned}
\int_k^{\infty}\frac{G'(p^*(v))F'(v)}{\psi'(p^*(v))}dv &= B^{-1}\int_{p^*(k)}^{\infty}\varphi(y)(yF(k^*(y))-rqk^*(y))m'(y)dy\\
&- B^{-1}F(k)\int_{p^*(k)}^{\infty}\varphi(y)ym'(y)dy - B^{-1}qk\frac{\varphi'(p^*(k))}{S'(p^*(k))}\\
&\leq B^{-1}\int_{p^*(k)}^{\infty}\varphi(y)\pi(y)m'(y)dy\\
&- B^{-1}F(k)\int_{p^*(k)}^{\infty}\varphi(y)ym'(y)dy - B^{-1}qk\frac{\varphi'(p^*(k))}{S'(p^*(k))},
\end{aligned}$$

where $\hat{k}(p) = \text{argmax}_k\{pF(k)-rqk\}$ and $\pi(p) = \sup_{k\in\mathbb{R}_+}[pF(k)-rqk]$ denotes the maximal short run profit flow. Since this flow was assumed to be integrable, we find that the integral expression measuring the value of future expansion options exist. Given this observation it is now from the proof of Theorem 2.1 that the value satisfies the variational inequalities $(\mathcal{A}V)(k,p)-rV(k,p)+pF(k)\leq 0$ and $V_k(k,p)\leq q$. Since this value is attained by applying an admissible policy, we find that the value reads as in (2.15). □

D Proof of Theorem 2.4

Proof. Equation (2.9) implies that on the continuation region $(0, p^*(k))$ we have

$$V_{kp}(k,p) = \psi'(p)\left[\frac{G'(p)}{\psi'(p)} - \frac{G'(p^*(k))}{\psi'(p^*(k))}\right]F'(k).$$

Since

$$\frac{G'(p)}{\psi'(p)} = B^{-1}\frac{\varphi'(p)}{\psi'(p)}\int_0^p \psi(y)ym'(y)dy + B^{-1}\int_p^\infty \varphi(y)ym'(y)dy$$

we find by ordinary differentiation that

$$\frac{d}{dp}\left[\frac{G'(p)}{\psi'(p)}\right] = \frac{2rS'(p)}{\sigma^2(p)\psi'^2(p)}\int_0^p (y-p)\psi(y)m'(y)dy < 0$$

implying that the marginal value is monotonically increasing as a function of the current price p on the continuation region $(0, p^*(k))$. Differentiating the marginal value twice with respect to the current price yields

$$V_{kpp}(k,p) = \left[G''(p) - \frac{G'(p^*(k))}{\psi'(p^*(k))}\psi''(p)\right]F'(k).$$

As was established in Alvarez (2003) the assumed monotonicity of the net appreciation rate $\mu(p) - rp$ implies that the increasing fundamental solution $\psi(p)$ is strictly convex on $\mathbb{R}_+$. Moreover, since

$$G'(p) = \mathbf{E}\int_0^\infty e^{\int_0^t(\mu'(p_s)-r)ds}M_t dt, \tag{D.1}$$

where

$$M_t = \exp\left(\int_0^t \sigma'(p_s)dW_s - \frac{1}{2}\int_0^t \sigma'^2(p_s)ds\right)$$

is a positive exponential martingale. Defining the equivalent measure $\mathbb{Q}$ by the likelihood ratio $d\mathbb{Q}/d\mathbb{P} = M_t$ implies that (D.1) can be re-expressed as

$$G'(p) = \mathbf{E}^{\mathbb{Q}}\int_0^\infty e^{\int_0^t(\mu'(p_s)-r)ds}dt.$$

The assumed concavity of the net appreciation rate implies that the $\mu'(p)$ is non-increasing and, therefore, that $G'(p)$ is non-increasing as well. Combining these observation imply that the marginal value is concave as a function of the current price p on the continuation region $(0, p^*(k))$. Since $V_k(k,p) = q$ on $[p^*(k), \infty)$ we find that the marginal value of capital is concave as a function of the current price p.

Denote as $\hat{V}_k(k,p)$ the marginal value of capital in the presence of the more volatile price dynamics $\hat{p}_t$ characterized by the differential operator

$$\hat{\mathcal{A}} = \frac{1}{2}\hat{\sigma}^2(p)\frac{\partial^2}{\partial p^2} + \mu(p)\frac{\partial}{\partial p}$$

where $\hat{\sigma}(p) \geq \sigma(p)$ for all $p \in \mathbb{R}_+$. It is now clear that the marginal value $\hat{V}_k(k,p)$ satisfies the inequality $\hat{V}_k(k,p) \leq q$ for all $(k,p) \in \mathbb{R}^2_+$. Moreover, since

$$(\mathcal{A}\hat{V}_k)(k,p) - r\hat{V}_k(k,p) - pF'(k) \geq \frac{1}{2}(\sigma^2(p) - \hat{\sigma}^2(p))\hat{V}_{kpp}(k,p) \geq 0$$

we find that the marginal value $\hat{V}_k(k,p)$ satisfies the sufficient variational inequalities and, therefore, is smaller than or equal to the marginal value of capital in the presence of the less volatile price dynamics p_t. That is, $\hat{V}_k(k,p) \leq V(k,p)$. Finally, denote as $C_{\hat{\sigma}} = \{(k,p) \in \mathbb{R}^2_+ : \hat{V}_k(k,p) < q\}$ the continuation region in the presence of the more volatile price dynamics $\hat{p}_t$ and as $C_\sigma = \{(k,p) \in \mathbb{R}^2_+ : V_k(k,p) < q\}$ the continuation region in the presence of the less volatile price dynamics p_t. If $(k,p) \in C_\sigma$ then $\hat{V}_k(k,p) \leq V_k(k,p) < q$ implying that $(k,p) \in C_{\hat{\sigma}}$ as well and, therefore, that $C_\sigma \subseteq C_{\hat{\sigma}}$. □

E Proof of Theorem 2.5

Proof. Assume that $p \in (a,b) \subset \mathbb{R}_+$ and that $0 < a < b < \infty$. It is known that the probability of hitting the upper boundary b before than the lower boundary a can be expressed as (cf. Borodin and Salminen (2002), p. 14)

$$\mathbb{P}_p[\tau_a > \tau_b] = \frac{S(p) - S(a)}{S(b) - S(a)},$$

where $S(p)$ denotes the scale function of p_t. Since 0 is a natural boundary for the underlying price process p_t we have that $\tau_0 = \infty$ a.s. Moreover, $\lim_{a \to 0} S(a) <$

∞ whenever 0 is attracting for the underlying price process p_t. Thus, in the case of our theorem

$$\mathbb{P}_p[\tau_b < \infty] = \frac{S(p) - S(0)}{S(b) - S(0)}.$$

However, since $\mathbb{P}_p[\mathcal{M}_t > z] = \mathbb{P}_p[\tau_z < t]$ for all $p < z$ we find that $\mathbb{P}_p[\hat{M} > z] = \mathbb{P}_p[\tau_z < \infty]$ for all $p < z$ (cf. Borodin and Salminen (2002), p. 26) and, therefore, that the optimal capital stock and marginal productivity of capital tend towards the proposed limits (2.16) and (2.17). $\square$

Aboa Centre for Economics (ACE) was founded in 1998 by the departments of economics at the Turku School of Economics and Business Administration, Åbo Akademi University and University of Turku. The aim of the Centre is to coordinate research and education related to economics in the three universities.

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