1   Introduction

The history of the Borda Count (BC, for brevity) is well-known. Introduced by Chevalier Jean-Charles de Borda to the Royal French Academy of Sciences in 1770 as a replacement of the then (and still now) widely used plurality voting, it met with a modicum of success in terms of practical application (in the French Academy of Sciences), but was soon largely forgotten to be rescued from oblivion about a hundred years later by E. J. Nanson and C.L. Dodgson.[1] It was not until Black's *magnum opus* (1958) in the end of 1950's that BC was brought to a comparative context with other social choice rules. It is fair to say that the going has been all but smooth for BC. The main criticism leveled against it today echoes the attack of Marquis de Condorcet and is based on a binary intuition of winning (see Risse 2005 for a recent criticism and Saari 2006 for its rebuttal).

In the following we shall first present some social choice criteria invoked in the debates on BC and see how the rule fares in terms of those criteria. We also evaluate an early modification of BC, Nanson's method. We then focus on another - strategic - set of performance criteria and assess BC in the light of these. Thereafter, we discuss a couple of recent competitors of BC as well as the modifications of BC itself.

2   Borda Count: the basic properties

BC is a point voting system where each voter provides a ranking. Each alternative is positioned in one and only one rank by each voter. In technical terms this means that each voter's vote expresses a complete and transitive preference order over the decision alternatives. Borda proposed that the lowest rank would be given a points, the next to lowest $a+b$ points, the next one $a + 2b$ points, and so on. In the preceding chapters, the values $a = 1$ and $b = 1$ have been applied. However, any other assignment of positive numbers will yield the same outcomes.The points given by each voter to an alternative are then summed up. This sum is called the Borda score of the alternative. The Borda winner is the alternative with the largest Borda score and the Borda ranking is the ordering of the alternatives consistent with their Borda scores, the larger the score, the higher the rank.

The main message of Borda's memoir presented to the French Academy was to show that the plurality voting - i.e. one person, one preference system - may lead to quite unacceptable outcomes. To wit, it may happen that the plurality winner would be regarded worse than any other alternative by

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[1]McLean and Urken (1995) give a thorough account of the history of BC arguing i.a. that the BC was actually invented several hundred years before Borda by Ramon Lull. Borda was, however, undeniably the first to discuss the method in any systematic and comparative detail. Hence, the nomenclature seems wholly appropriate.

4 voters	3 voters	2 voters
A	B	C
B	C	B
C	A	A

Table 1: BC vs. plurality voting

4 voters	2 voters	1 voter
A	B	C
B	C	B
C	A	A

Table 2: BC vs. Condorcet winner

a majority of voters in pairwise contests. A slightly simplified version of Borda's example is presented in Table 1.

The table expresses the preferences of voters in the customary way. E.g 4 voters rank A before B and B before C. Assuming that these preferences are revealed in the ballots cast, we observe that A would win the plurality contest by 4 votes against B's 3 and C's 2 votes. And yet, A would be defeated by both B and C in pairwise majority comparisons; B would win A with 5 votes against 4 and C would also win A by the same margin. Hence, the case for arguing that A expresses the will of the collective body is very weak, indeed. In modern terminology alternatives that are defeated by all others by majority in pairwise contests are called Condorcet losers. It is clear that Borda wanted to avoid such alternatives being chosen. This was the main point of his criticism against plurality voting. One of the the primary virtues of his proposal, BC, is to exclude the possibility of the choice of an eventual Condorcet loser.

Of the other theoretical properties of BC perhaps the best-known is the possibility that it may not elect a Condorcet winner, i.e. an alternative that in pairwise contests would defeat all others in pairwise comparisons. In Table 1 B is the Condorcet winner since it beats A with 5 votes against 4 and C with 7 votes to 2. In this example, however, BC happens to elect the Condorcet winner. Modifying the example slightly so as to get Table 2 we see, however, that BC may not end up with the Condorcet winner ranked first. A is clearly the Condorcet winner, but B gets the highest Borda score.

In Table 2 the discrepancy between the Condorcet winning criterion and BC is particularly marked since A is the strong Condorcet winner, i.e. is top-ranked by more than 50% of the voters.

The critics of BC have pointed to the discrepancy between BC outcome and the Condorcet winner as the main flaw of BC. Some - e.g. Riker (1982)

2 voters	2 voters	2 voters	1 voter
C	A	B	C
B	C	A	B
A	B	C	A

Table 3: BC and deleted alternatives

- have also called attention to the "instability" of BC rankings under modifications - expansions or subtractions - of the alternative set (Fishburn 1974; Hill 1988). The following example illustrates (Nurmi 1998, 126).

In Table 3 the Borda ranking is CBA. Suppose, however, that B, for some reason, is not available. Deleting B and recomputing the Borda scores for the remaining alternatives yields the Borda ranking AC. This is a reversal of the ranking over these three alternatives in the original setting. Thus, among A and C, C is the winner if B is present, but A is the winner if B is absent. Hence, the Borda winner in the set X is not necessarily the Borda winner in all proper subsets of X containing it. In fact, Fishburn's result states that the Borda winner in a set X of alternatives is not necessarily the Borda winner in any proper subset of X except one. In other words, if an alternative is the Borda winner in a set consisting of 8 candidates, it has to be the Borda winner in no more than one out of the 127 coalitions it is a member of.

3 Early remedy?

About a hundred years after the publication of Borda' memoir, E. J. Nanson (1883) published a systematic comparison of a variety of voting systems. In contrast to his predecessors in the theory of voting, Nanson was well aware of the major developments in his field. In particular, he knew that the Condorcet winner is not necessarily elected by BC. He set out to devise a modification of BC that did not have this flaw. Nanson's proposal - today called Nanson's method - is based on the observation that despite the fact that the Condorcet winner does not necessarily receive the highest Borda score, there is a connection between the Condorcet winner and the Borda scores. To wit, the Condorcet winner never gets the lowest Borda score. In fact, it can be shown that the Condorcet winner always gets a strictly higher than average Borda score.

These observations led Nanson to suggest that BC be used as an elimination device so that at each stage of the process, those alternatives with at most the average Borda score are eliminated. After the elimination, new Borda scores are computed for the remaining alternatives disregarding the eliminated ones. The process is repeated until we are left with a unique win-

5 voters	9 voters	5 voters	9 voters	13 voters	2 voters
B	B	A	A	C	A
A	D	C	B	A	C
C	C	B	D	D	D
D	A	D	C	B	B

Table 4: Nanson's method is non-monotonic

ner or a tie between some alternatives. The elimination criterion guarantees that if there is a Condorcet winner, it will not be eliminated. Thus, it will be elected by Nanson's method.

Nanson was, thus, able to secure the satisfaction of both Condorcet criteria - viz. that an eventual Condorcet winner be elected and that the eventual Condorcet loser not be chosen - by a method that is very much in the spirit of BC. However, the cost of securing the former criterion is high: Nanson's method is non-monotonic. In other words, additional support may turn a winning candidate into a losing one. This is illustrated by Table 4.

Here A becomes the Nanson winner after first D and both B and C are eliminated. Suppose now that the left-most group of 5 voters changes its mind in A's favor so that its ranking is ABCD. Nanson's method results in the new profile in C. Thus, the winner A's additional support renders it non-winner. This shows that Nanson's method is non-monotonic. The price of Condorcet consistency, that is, choosing a Condorcet winner when one exists, thus seems to be the loss of monotonicity. This price is perhaps too high. Hence the question mark in the section heading.

4 Incomplete ballots: Borda's forte lost

When the number of candidates or policy alternatives considered is large, it is unreasonable to expect the voters to rank each and every one of them. Yet, BC in its basic form requires this. So, what to do if a person simply ranks a couple of his/her most preferred options, but leaves the rest blank? It would seem reasonable to allow this type of behavior. Hence, the most voter-hostile way of proceeding - which is to disqualify such ballots - seems indefensible. After all, the voter has clearly expressed his/her ranking over a few alternatives. Moreover, it is plausible to assume that he/she has also provided a ranking between these alternatives, on the one hand, and those not ranked, on the other. He/she obviously prefers each ranked alternative to each not ranked one. But how to assign points to alternatives in this setting?

Several ways of proceeding can be envisioned:

1. Assume that incomplete ballots indicate a tie between all alternatives

that have not been ranked and compute the Borda scores accordingly. For example, if there are 5 alternatives A, B, C, D, E and a voter ranks A before B leaving other alternatives unranked, we assign A five, B four and C, D, E, the average score, i.e. two points, each. This would give each voter the same number of points to be distributed regardless of whether he/she ranks all the alternatives.

2. Assume that all unranked alternatives are, in fact, ranked last, i.e. given zero points by the voter casting an incomplete ballot, and the ranked ones are given Borda scores as if all alternatives were ranked. In the above 5-alternative example, A would be given five, B four and C, D, E zero points each. This would allow the voter to make a disproportionately large difference between his/her favorites and the unranked alternatives. In fact, this would encourage strategic behavior since by casting an incomplete ballot, a voter may increase the score difference between his/her favorites and the other alternatives from the difference he/she would be able to make by casting a complete ballot.

3. The modified BC (MBC). This system, elaborated in the preceding chapters, reduces the Borda point of the first ranked alternative by 1 for every unranked alternative. In the preceding example, A would receive two and B one points, while C, D, E get zero points each. Thus, the strategic incentives for preference "truncation" are smaller than in systems described above.

4. The quota Borda system (QBS). This differs from the preceding one in introducing a new criterion for election of a candidate, viz. quota q. This is obtained by dividing the number or voters by the number of vacancies to be filled plus one. Any candidate who is ranked first by at least q voters is elected. In constituencies sending $4-5$ representatives, also any pair of candidates which has been ranked first or second by at least $2q$ voters is elected. Analogously, in constituencies sending at least 6 representatives, any triplet of candidates ranked first, second or third by at leat $3q$ voters is elected.

The main motivation of these modifications is to handle incomplete ballots. These, in turn, are likely to be encountered in elections with large number of candidates. It turns out, however, that neither MBC nor QBS guarantees the exclusion of an eventual Condorcet loser. The following example illustrates this.

In Table 5 A is the Condorcet loser since it is defeated by all other alternatives in pairwise comparisons with 5 votes to 4. Alternative B gets the highest Borda score.

4 voters	3 voters	1 voter	1 voter
A	B	C	E
B	E	D	C
C	D	B	D
D	C	E	B
E	F	G	F
F	G	F	G
G	A	A	A

Table 5: MBC and QBS may elect a Condorcet loser

Suppose that the three rightmost groups consisting of 3, 1 and 1 voters, respectively, cast an incomplete ballot indicating only their first preference, while the leftmost group indicates its entire ranking. Under these circumstances, the MBC elects A, the Condorcet loser. Assume that four representatives are elected. Then also QBS includes the Condorcet loser A, since it, along with B, exceeds the quota 2. Thus QBS elects the Condorcet loser.

It can be shown that modifications 1 and 2 in the above list can also end up with a Condorcet loser. Thus, all BC modifications of this section result in the choice of the Condorcet loser. The price for accommodating incomplete ballots with the above techniques may seem high.

5   How robust are the Condorcet winners?

The fact that BC may not elect the Condorcet winner is often deemed its main flaw. The importance of the Condorcet winning criterion is visible in those incompatibility theorems where this criterion is shown to be incompatible with this or that desirable property of the choice rule. For example, there is a theorem showing the incompatibility between the Condorcet winning criterion and invulnerability to the no-show paradox (Moulin 1988). Another one relates the Condorcet criterion with manipulability of choice functions (Gärdenfors 1976). Clearly, the significance of these and other related results is the greater, the more compelling is the Condorcet winning criterion. Yet, an argument can be built to the effect that the criterion is not all that compelling. Let us look at Fishburn's (1973) example (Table 6).

Here D is the Condorcet winner, but one could make a strong case for electing E. To wit, E is ranked first by as many voters as D, E is ranked second by more voters than D and E is ranked third by more voters than D. Moreover, E has now lower ranks in any voter's preference, while D is ranked once fourth and once last. Surely, E would seem more plausible choice than the Condorcet winner D. Now, why is that? The reason is simply that we know the entire preference rankings of all voters and in the light of those

1 voter	1 voter	1 voter	1 voter	1 voter
D	E	C	D	E
E	A	D	E	B
A	C	E	B	A
B	B	A	C	D
C	D	B	A	C

Table 6: Implausibility of the Condorcet winner

5 voters	3 voters
A	B
B	C
C	A

Table 7: BC and (strong) Condorcet winner

rankings E seems to be ranked higher on the average than D. There is a technical word - coined by Fishburn (1982) - for this type of superiority: positional dominance. An alternative x positionally dominates alternative y if x has at least as many first ranks as y, as many first or second ranks as y, etc. until the penultimate rank. We see that positional dominance may contradict pairwise majority voting.

The case for the Condorcet winning criterion becomes even more contestable once we see that Condorcet winners are surprisingly unstable under modifications of preference profiles through adding or subtracting groups of voters. Saari (1995) shows that a group of voters whose preferences over alternatives form an instance of the Condorcet paradox, when added to an existing preference profile, may "destroy" a Condorcet winner. An yet, the Condorcet paradox is a completely symmetrical setting where an equal number of voters rank each of the three alternatives first, second and third. To illustrate, consider the following setting (Table 7) (Nurmi 2002, 124-126).

Table 7 exhibits a dramatic instance of the discrepancy between the Condorcet and Borda winners. A is the Condorcet winner, indeed, a strong one in the sense of being ranked first by a majority of voters. B, on the other hand, is the Borda winner.

Consider now an instance of the Condorcet paradox shown in Table 8. Unless one treats voters or alternatives in some discriminating fashion, there is no way of telling which of the three alternatives should be elected. The setting is a perfect tie. So, we have two settings: Table 7 where the choice is clear, on the one hand, and Table 8 where no alternative should be preferred to the others, on the other. Now, adding the voters of Table 8 to the profile of Table 7 should, intuitively, leave the winner of the latter profile intact.

3 voters	3 voters	3 voters
A	B	C
C	A	B
B	C	A

Table 8: The Condorcet paradox

And it does, if one applies BC. However, if one resorts to any Condorcet extension method - i.e. a method that always results in the choice of the Condorcet winner when one exists - the outcome changes from A to B, the new Condorcet winner. So, adding a group of voters whose preferences form a tie, changes the outcome of Condorcet extension methods.

Saari also shows that nearly all positional voting procedures - e.g. plurality voting and anti-plurality voting - are sensitive to adding or subtracting voting groups of equal size but with diametrically opposed preferences such as ABC and CBA. Of positional procedures only BC is invulnerable to these kinds of changes. To illustrate, add 3 voters with preference CAB and 3 voters with opposing preference BAC to the Table 7 profile where A is the plurality winner. It turns out that in the resulting profile, B emerges as the plurality winner.

Quite a strong case can thus be made for BC as it is the only procedure that leaves the winners intact after adding Condorcet paradox groups or groups with preferences that "cancel out" each other.

6   Ways out of majority tyranny

One of the perennial problems of constitutional design is to avoid permanent majorities to exploit the minorities without slipping into the rule of minority. The standard way of handling the problems of majority tyranny - or more neutrally expressed, majority decisiveness - is to impose high majority thresholds for proposals to pass in the collective decision making body. The extreme case is, of course, the rule of unanimity which *ipso facto* guarantees that no one objects the decisions passed by the body. This rule has an unpleasant feature, though: any voter can veto any proposal. Hence, the rule has a very strong *status quo* bias.

In the other end of the majority rule spectrum is the simple majority principle which states that if a proposal is backed by strictly more than 50% of the voters, it will pass. Hence, any majority can dictate the decision outcomes. Baharad and Nitzan (2002) prove an interesting result relating the majority threshold to the type of point voting system. Point voting systems are methods based on individual preference rankings where $p(1), \ldots, p(k)$ are the points assigned to alternatives ranked first, $\ldots, k$th. In plurality voting

7 voters	7 voters	3 voters	3 voters
A	B	C	C
D	C	D	D
C	D	B	A
B	A	A	B

Table 9: QBS and BC

$p(1) = 1$ and $p(2) = \ldots = p(k) = 0$, while in BC $p(1) = k, \ldots, p(k) = 1$. Baharad and Nitzan's result states that when the voters vote sincerely and with majority threshold $\alpha, (0 \leq \alpha \leq 1)$, the condition

$$\alpha(p(k) - p(k-1)) < (1-\alpha)(p(k) - p(1))$$

implies that the point voting system is immune to majority decisiveness. Since in BC this expression reduces to

$$k > \frac{1}{1-\alpha}$$

even a modest number of alternatives is sufficient to rule out tyranny of even large majorities. For example, 4 alternatives guarantees immunity to decisiveness of up to 3/4 majority. This result pertains to sincere voting. When the voters can coordinate their voting strategies, BC is guarantees immunity to all $\alpha-$majority rules when $\alpha \leq 2/3$ (Baharad and Nitzan 2002). BC provides thus a fairly strong protection against majority tyranny.

What happens when we institute a quota in accordance with QBS? In contrast to what one would expect, the effect of the quota is to lower the threshold for majority decisiveness. To wit, by setting the quota at $n/(e+1)$, where n is the number of voters and e the number of elected candidates, one often enables smaller majorities to be decisive than would be the case if BC were applied. Stated in another way: in QBS one needs the coordination of larger groups to gain representation than in BC. Thus, QBS is less minority empowering than BC. Consider the example of Table 9.

Assuming that two representatives are to be elected, BC ends up with C and D. On the other hand, since the quota is 7, A and B are elected under QBS. The outcome of QBS remains the same if we assume that the two leftmost voter groups reveal only their first preference. It seems, then, that QBS is considerably more majoritarian in spirit than BC. This is a characteristic of QBS that becomes more visible when only one candidate is to be elected. For example, in Table 7 QBS results in the strong Condorcet winner A, while BC elects B.

	East			West	
5	3	1	4	3	3
A	B	C	A	B	C
B	C	B	B	C	A
C	A	A	C	A	B

Table 10: QBS is inconsistent

7 A word on consistency

One of the virtues of BC is consistency (Young 1974). In social choice theory this concept is defined as follows. Suppose that two distinct groups of voters - say, two municipalities that together form a constituency - are electing a representative body from the same set of candidates using the same voting procedure. Suppose, moreover, that these groups end up with at least partially overlapping set of representatives. Now, the procedure used by both groups is said to be consistent, if under these circumstances the procedure, when applied to the ballots of both groups simultaneously, results always exactly in the overlapping set of winners. In single-winner elections consistency means that if candidate x wins in both municipalities, he/she always wins also in the constituency as a whole. Despite its intuitive plausibility, consistency is not a common property among voting systems (see, e.g. Nurmi 1987, 92-107). However, BC is consistent. QBS, on the other hand, is not. The example of Table 10 illustrates.

The single-member constituency consists of East and West, the former with 9 and the latter with 10 voters. Applying QBS, A exceeds the quota in East and is elected, while in West it wins because it has the highest Borda score. Combining the ballots of East and West, we observe that no candidate exceeds the quota and hence the winner is determined by the Borda scores. Now, it turns out that B wins. Hence, QBS is inconsistent.

8 Comments on the matrix vote

The matrix vote enables the voters to indicate their preference simultaneously over candidates and positions. More precisely the voter is able to indicate his/her preference over "states of affairs" where each state is a combination of a candidate and the office. One state of affairs could be, e.g. "candidate J in position A". This, of course, tells something different of voter preferences than the assignment of Borda points to candidates (or, for that matter, to positions). At the same time it should be observed that the voters are not able to disclose very much information about their views. Namely, for each position or office the voter can only reveal his/her favourite

candidate precisely as in plurality voting. What distinguished the matrix vote from the plurality voting is that the voters also reveal their priorities regarding the positions. In other words, the matrix vote is a mixture of plurality and BC. The voters may rank the positions, but not the candidates to each position (except in the limited sense of indicating their first preference).

The matrix vote outcomes are determined on the basis of the MBC scores. For example in Table 3.W the first position to filled is that of minister of B since the row corresponding to that ministerial post has the largest entry in the whole Table. The next posts are filled applying the same principle. Invoking the just mentioned interpretation of the scores, it can be said that the state of affairs where candidate Q becomes the minister of B has the largest collective preference among single ministerial post allocations.

This seems a plausible way to determine the composition of the cabinet. At least it provides the voters an incentive to think not only about their favourite candidates, but also about which tasks those candidates would be best suited. On closer inspection, though, Table 3.W reveals something of an anomaly, viz. Mr O becomes the minister of D and, yet, not a single voter has given him a single point for that ministerial portfolio. What we have here is a variation of the paradox of multiple elections discussed by Brams et al. (1997; 1998). When a ballot is taken separately on several policy issues, the (majority) winning combination of policies may be one that was supported by no voter. Here the portfolio allocation over candidates is one that is supported by not a single voter.

The matrix vote combines preferential information of candidates with that of positions. As stated above, the result is a mixture of plurality voting and BC. The end result may, thus, grossly deviate from a position-by-position BC. In other words, it the voters were allowed to indicate their preference rankings over candidates for each position separately, the portfolio allocation could be very different from the one resulting from the matrix vote. This is not surprising since differences between BC and plurality voting are due to the fact that the former utilizes the preference information to a far larger extent than the latter. The matrix vote has one advantage over the position-wise BC, viz. it is relatively easy to implement. At the same time it allows - albeit in a very restricted manner - the expression of two types of preference information: one pertaining to candidates and the other to positions.

9   Conclusion

The discrepancy between BC and the requirement that the Condorcet winner be elected whenever one exists is well known from the early days of the social choice theory. Nanson's method sets out to remove this discrepancy

by eliminating alternatives and repeating BC, while simultaneously making sure that the eventual Condorcet winner is not eliminated on the way. The price of achieving compatibility is, however, high: Nanson's method is non-monotonic. More recent variations of BC - MBS and QBS - aim at allowing for incomplete ballots which are bound to become increasingly common in large sets of candidates. It turns out that while both of the modifications are monotonic, they may include a Condorcet loser in their choice sets. It is also possible that while in general successful in protecting minority opinions, QBS sometimes also leads to a more majoritarian outcome than BC.

References

Baharad, E. and Nitzan, S. (2002), Ameliorating Majority Decisiveness through Expression of Preference Intensity, *American Political Science Review 96*, 745-754.

Brams, S., Kilgour, D. M. and Zwicker, W. (1997), Voting on Referenda: The Separability Problem and Possible Solutions, *Electoral Studies 16*, 359-377.

Brams, S., Kilgour, D. M. and Zwicker, W. (1998), The Paradox of Multiple Elections, *Social Choice and Welfare 15*, 211-236.

Fishburn, P. (1973), *The Theory of Social Choice*, Princeton: Princeton University Press.

Fishburn, P. (1974), Voting Paradoxes, *American Political Science Review 68*, 537-546.

Fishburn, P. (1982), Monotonicity Paradoxes in the Theory of Elections, *Discrete Applied Mathematics 4*, 119-134.

Gärdenfors, P. (1976), Manipulation of Social Choice Function, *Journal of Economic Theory 13*, 217-228.

Hill, I. (1988), Some Aspects of Elections - to Fill One Seat or Many, *Journal of the Royal Statistical Association, Series A, 151*, 243-275.

McLean, I. and Urken, A., (eds) (1995), *Classics of Social Choice*, Ann Arbor: The University of Michigan Press.

Moulin, H. (1988), Condorcet's Principle Implies the No Show Paradox, *Journal of Economic Theory 45*, 53-64.

Nanson, E. J. (1883), Methods of Election, *Transactions and Proceedings of the Royal Society of Victoria XIX*, 197-240. (Reprinted in McLean and Urken (1995)).

Nurmi, H. (1998) *Rational Behaviour and the Design of Institutions*, Cheltenham: Edward Elgar.

Nurmi, H. (2002), *Voting Procedures under Uncertainty*, Berlin-Heidelberg-New York: Springer-Verlag.

Riker, W. (1982) *Liberalism against Populism*, San Francisco: W. H. Freeman.

Risse, M. (2005), Why the Count de Borda Cannot Beat the Marquis de Condorcet, *Social Choice and Welfare 25*, 95-113.

Saari, D. (1995), *Basic Geometry of Voting*, Berlin-Heidelberg-New York: Springer Verlag.

Saari, D. (2006), Which Is Better: The Condorcet or Borda Winner?, *Social Choice and Welfare 26*, 107-129.

Young, H. P. (1974), An Axiomatization of Borda's Rule, *Journal of Economic Theory 9*, 43-52.