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ABSTRACT

An undirected connected bimodal network has two Bonacich measures quantifying the centrality of the nodes. We show that the product of Bonacich measures of an undirected bimodal network may be viewed as a product measure that is nearest (w.r.t. Euclidean norm) to the matrix representing the network. A directed bimodal network has four Bonacich measures, two inflow and two outflow measures. Given directed strongly connected bimodal network and its Bonacich inflow (or outflow) measures, there is undirected bimodal network with the same Bonacich measures. So Bonacich measures of directed bimodal network can also be viewed as minimum norm solutions.

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1. Introduction

An undirected connected bimodal network has two Bonacich measures quantifying the centrality of the nodes. We show that the product of Bonacich measures of an undirected bimodal network may be viewed as a product measure that is nearest (w.r.t. Euclidean norm) to the matrix representing the network. A directed bimodal network has four Bonacich measures, two inflow and two outflow measures. Given directed strongly connected bimodal network and its Bonacich inflow (or outflow) measures, there is undirected bimodal network with the same Bonacich measures. So Bonacich measures of directed bimodal network can also be viewed as minimum norm solutions.

The node set of a bimodal network is partitioned into two disjoint subsets, and all the links are between these subsets. Bonacich (1991) introduced measures that can be used to quantify the importance of agents in a bimodal network. A canonical interpretation of the model is that there are n individuals and m clubs, and each individual is a member of at least one club. The strength of a link between individual i and club j tells how strong is the connection between i and j.

Bonacich studied explicitly undirected and unweighted bimodal networks. In such a case, a $m \times n$ matrix whose elements are zeroes and ones fully represents the network. His analysis extends directly to weighted networks. Such networks are representable by nonnegative $m \times n$ matrices.

From a bimodal network it is possible to construct two unimodal networks with individuals and clubs as nodes, respectively (details in the next section). The eigenvector centrality measures (see *e.g.* Jackson and Zenou 2014) of these networks are actually the Bonacich measures of the original bimodal network.

A directed bimodal network may be represented by two $m \times n$ matrices. One of the matrices gives the strenghts of links from the individual to clubs, and the other matrix gives the strenghts of links to the opposite direction. Again, the analysis and definition of Bonacich measures follows the ideas given in Bonacich (1991). Salonen (2015) shows that in this case these measures can be interpreted as Nash equilibria of certain noncooperative games.

Notation and definitions are given in Section 2. The main results are stated and proven in Section 3.

2. Preliminaries

Let G be a bimodal undirected network with vertex set $V_1 \cup V_2$ such that V_1 and V_2 are disjoint and $|V_1| = m$, $|V_2| = n$. Such a network can be represented by a nonnegative $m \times n$ matrix A, whose element a_{ij} gives the strength of the link between $i \in V_1$ and $j \in V_2$. If $a_{ij} = 0$ then there is no link between i and j. There exists a path between nodes $i, j \in V_1 \cup V_2$, if there exists a sequence of strictly positive links connecting i and j.

Network G is *connected*, if a path exist between all $i, j \in V_1 \cup V_2$, $i \neq j$. In such a case both matrices $A^T A$ and AA^T are *irreducible*: for any $i, j \in V_1$ (respectively $i, j \in V_2$) there is some natural number $k > 0$ such that the ij -element of $(A^T A)^k$ (respectively $(AA^T)^k$)

is strictly positive. Note that $A^T A$ represents a unimodal network on the node set V_1, and AA^T represents a unimodal network on the node set V_2.

Bimodal networks are assumed to be connected in in this paper. Then there exist strictly positive α, β such that

$$Ay = \alpha x \tag{1}$$
$$x^T A = \beta y^T, \tag{2}$$

for some strictly positive vectors x and y which are called the Bonacich measures of the bimodal network (see Bonacich 1991). Vector x is a column vector, and x^T is its transpose. Inserting x from equation (1) in equation (2), and then y back into equation (1) we get:

$$AA^T x = \alpha\beta x \tag{3}$$
$$A^T Ay = \alpha\beta y. \tag{4}$$

The eigenvalue $\alpha\beta$ is the largest eigenvalue of these symmetric matrices, and x and y are strictly positive vectors.

Let $\|\cdot\|$ be the Euclidean norm. The spectral norm of a matrix M is defined by

$$\|M\|_s = \max \|Ax\|, \ \ s.t. \ \|x\| \leq 1.$$

By equation (3), $\|A\|_s = \sqrt{\alpha\beta}$.

Let $G^{\uparrow}$ be a directed bimodal network on the node set $V_1 \cup V_2$. Such a network can be represented by nonnegative $m \times n$ matrices A and B (see Salonen 2015). The element a_{ij} gives the strength of the link from node $i \in V_1$ to node $j \in V_2$, and element b_{ij} gives the strength of the link from node $j \in V_2$ to node $i \in V_1$. Network $G^{\uparrow}$ is strongly connected if for any distinct nodes i and j, there is a path leading from i to j and *vice versa.* This means that both matrices AB^T and $B^T A$ are irreducible.

Note that AB^T represents a unimodal directed network on the node set V_1. The strength of the link from $i \in V_1$ to $j \in V_1$ is given by $\sum_{k \in V_2} a_{ik} b_{jk}$. Similarly, $B^T A$ represents a unimodal directed network on the node set V_2. The strength of the link from $i \in V_2$ to $j \in V_2$ is given by $\sum_{k \in V_1} b_{ik} a_{kj}$.

The left eigenvectors y of $B^T A$ and x of AB^T are related in the following way. Consider the equations.

$$By = \alpha x \tag{5}$$
$$x^T A = \beta y^T, \tag{6}$$

for some $\alpha, \beta > 0$. Now $By = \alpha x$ means that $y^T B^T = \alpha x^T$. Hence $x^T = (1/\alpha) y^T B^T$, and therefore from equation (6) we get that $(1/\alpha) y^T B^T A = \beta y^T$. On the other hand, from equation (6) we get that $y = (1/\beta) A^T x$, and then equation (5) implies $BA^T x = \alpha\beta x$. We have the following equations

$$y^T B^T A = \alpha\beta y^T \tag{7}$$
$$x^T AB^T = \alpha\beta x^T. \tag{8}$$

The number y_j (x_i) measures the importance of the node $j \in V_2$ ($i \in V_1$) in terms of inflow. The right eigenvectors of $B^T A$ and AB^T would give corresponding measures in terms of outflow. These are the four Bonacich measures of a directed bimodal network (see Salonen 2015).

3. Results

Let A be a nonnegative $m \times n$ matrix representing a connected bimodal network G. We want to find vectors $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_n)$ such that the tensor product pq^T solves the following norm minimization problem:

$$\min_{p,q} \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} (p_i q_j - a_{ij})^2, \tag{9}$$

where a_{ij} is the ij'th cell of A. Note that since p and q can be interpreted as being nonnegative measures, the matrix pq^T can be thought as representing their product measure $p \times q$. The norm in problem (9) is sometimes called the *Frobenius norm.* But if we view matrices as "long vectors" then it is actually the usual Euclidean norm.

The first order conditions are the following

$$p_i \sum_j q_j^2 - \sum_j a_{ij} q_j = 0, i = 1, \ldots, m \tag{10}$$

$$q_j \sum_i p_i^2 - \sum_i a_{ij} p_i = 0, j = 1, \ldots, n. \tag{11}$$

In matrix form the first order condition can be written a

$$(q^T q)p - Aq = \mathbf{0} \tag{12}$$

$$(p^T p)q - A^T p = \mathbf{0}, \tag{13}$$

where $\mathbf{0}$ is a vector of zeros. From these equations we get the following.

$$AA^T p = (p^T p)(q^T q)p, \tag{14}$$

$$A^T A q = (p^T p)(q^T q)q. \tag{15}$$

Now take the eigenvalue $\alpha\beta$ of the matrix AA^T, and choose the eigenvector x of AA^T such that $x^T x = \beta$. Similarly, take the eigenvalue $\alpha\beta$ of the matrix $A^T A$, and choose the eigenvector y of $A^T A$ such that $y^T y = \alpha$. Then these eigenvectors satisfy the first order conditions (12) and (13).

Note that any pair of eigenvectors x and y satisfies these first order conditions as long as they satisfy $(x^T x)(y^T y) = \alpha\beta$. In other words, their norms $\|x\| = \sqrt{\beta'}$ and $\|y\| = \sqrt{\alpha'}$ must satisfy $\sqrt{\alpha'}\sqrt{\beta'} = \sqrt{\alpha\beta}$, where $\sqrt{\alpha\beta}$ is the spectral norm of the matrices A and A^T. Note also that such eigenvectors x and y together with numbers β' and α' will satisfy equations (1) and (2).

We have the following.

Theorem 1. *Let A be a nonnegative $m \times n$ matrix representing a connected bimodal network G, and let x and y be a pair of Bonacich measures of this network. The product measure $x \times y$ minimizes the Euclidean norm $\|p \times q - A\|$ among all strictly positive product measures $p \times q$, if and only if the product $\|x\|\|y\|$ of their norms is the spectral norm $\|A\|_s$ of the matrix A.*

Given a matrix A that represents a bimodal network, the Bonacich measures x and y can be rescaled by multiplying by positive constants t and t', and the numbers $t, t' > 0$ may be chosen independently of each other. If x and y are chosen in such a way that their product minimizes a norm like in Theorem 1, then the numbers t and t' must satisfy $tt' = 1$, or $t' = 1/t$. Otherwise the condition between the norms of the Bonacich measures and the norm of the matrix A does not hold.

The norm condition given in Theorem (1) may be inconsistent with both x and y being probability measures. However, the Bonacich measures of a bimodal network are scale invariant in the sense that the Bonacich measures will not change if the matrix A representing the network is multiplied by a positive constant $t > 0$. The norm $\|A\|_s$ will of course be changed by the same scalar, and hence the eigenvalues of AA^T and A^TA will be changed to $t^2\alpha\beta$.

We have the following.

Theorem 2. *Let A be a nonnegative $m \times n$ matrix representing a connected bimodal network G, and let x and y be a pair of Bonacich measures of this network such that $\sum_i x_i = 1$ and $\sum_j y_j = 1$. Let $t > 0$ be a such that $\|tA\|_s = \|x\|\|y\|$. Then the product probability measure $x \times y$ minimizes the Euclidean norm $\|p \times q - tA\|$ among all strictly positive product probability measures $p \times q$.*

Let now A and B be $m \times n$ matrices representing a *directed* strongly connected bimodal network $G^{\uparrow}$. The element a_{ij} gives the strength of the link from node $i \in V_1$ to node $j \in V_2$, and element b_{ij} gives the strength of the link from node $j \in V_2$ to node $i \in V_1$. The left and right eigenvectors of B^TA and AB^T give the four Bonacich measures of this network as explained in Section 2.

We will assume that $\alpha = \beta = 1$ in equations (7) and (8), so that the greatest eigenvalues of both matrices B^TA and AB^T are equal to 1. This can be achieved by multiplying matrices A and B by suitable positive numbers, since B^TA has at least one strictly positive eigenvalue. Note that such a normalization does not change the Bonacich measures of the network.

Suppose now that B^TA has r strictly positive eigenvalues $1 = \gamma_1 \geq \cdots \geq \gamma_r > 0$. Recall that AB^T has these same strictly positive eigenvalues.

Consider an $m \times n$ matrix C whose singular value decomposition is

$$C = X\Gamma Y^T, \tag{16}$$

where Γ is an $r \times r$ matrix with diagonal elements $\Gamma_{11} = \sqrt{\gamma_1} = 1, \ldots, \Gamma_{rr} = \sqrt{\gamma_r}$, X is an $m \times r$ matrix whose columns are the left eigenvectors x_i of AB^T corresponding to the eigenvalues γ_i, and Y is an $n \times r$ matrix whose columns y_i are the left eigenvectors y_i of

$B^T A$ corresponding to the eigenvalues γ_i. These eigenvectors are assumed *w.l.o.g.* to be orthogonal: $x_i^T x_i = 1$ and $x_i^T x_j = 0$, for all $i, j = 1, \ldots, r$, $i \neq j$, and analogously for y_j.

Now $C^T C = Y\Gamma^2 Y^T$, and $C^T C y_i = \gamma_i y_i$, $i = 1, \ldots, r$. Similarly, $CC^T = X\Gamma^2 X^T$, and $CC^T x_i = \gamma_i x_i$, $i = 1, \ldots, r$. Since $C^T C$ is positive semidefinite, all its eigenvalues are non-negative, and therefore it has the same positive eigenvalues and corresponding eigenvectors as $B^T A$. Similarly, CC^T and AB^T have the same positive eigenvalues and corresponding eigenvectors.

Consider the following minimization problem.

$$\min_{p,q} \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} (p_i q_j - c_{ij})^2, \tag{17}$$

where c_{ij} is the ij -element of the matrix C. This problem is formally equivalent to problem (9), and hence the first order conditions will be similar as well. The solutions p and q must therefore satisfy the following equations

$$CC^T p = (p^T p)(q^T q) p, \tag{18}$$

$$C^T C q = (p^T p)(q^T q) q. \tag{19}$$

But we have just shown that the left eigenvectors $p = x_1$ and $q = y_1$ of AB^T and $B^T A$, respectively, satisfy these equations since $x_1^T x_1 = 1$ and $y_1^T y_1 = 1$.

The matrix C can be thought as representing an undirected bimodal network G with node sets V_1 and V_2. Then CC^T and $C^T C$ would represent undirected unimodal networks on the node sets V_1 and V_2, respectively. We may interpret C as the best approximation of the pair (A, B), among all $m \times n$ matrices that represent undirected bimodal networks with node sets V_1 and V_2.

We have the following result.

Theorem 3. *Let x and y be the Bonacich measures of a bimodal directed and strongly connected network $G^\uparrow$ on the node set $V_1 \cup V_2$, represented by matrices A and B in equations (7) and (8) with $\alpha = \beta = 1$. Then there is an undirected bimodal network G on the same node set, represented by a matrix C of equation (16), such that x and y are the Bonacich measures of G.*

If the directed network $G^\uparrow$ is such that matrices $B^T A$ and AB^T are symmetric, then the undirected network G represented by the matrix C approximates $G^\uparrow$ in a very precise sense as we shall see now. Consider the following norm minimization problem:

$$\min_{M} \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (M_{ij} - D_{ij})^2, \tag{20}$$

where M_{ij} is a typical element of a symmetric positive semidefinite $n \times n$ matrix M, and D is any real $n \times n$ matrix. In words, we try to find a symmetric positive definite matrix

M that minimizes the Frobenius norm $\|M - D\|$. Higham (1988) shows that the solution to problem (20) is

$$M = Z\Lambda_+ Z^T, \tag{21}$$

where Z is an orthgonal matrix and Λ_+ is a diagonal matrix. More concretely, note that D can be written as an average of its symmetric and skew-symmetric parts: $D = (D+D^T)/2 + (D - D^T)/2$. The symmetric part has the spectral decomposition $(D + D^T)/2 = Z\Lambda Z^T$, where Λ is the diagonal matrix whose diagonal elements are the eigenvalues of D, and Z is the matrix whose columns are the corresponding eigenvectors that are chosen to be orthogonal. Let Λ_+ be the diagonal matrix that is otherwise the same as Λ except that negative eigenvalues have been replaced by zeros.

Proposition 1. *Suppose A and B are such that B^TA and AB^T are symmetric. If $D = B^TA$ in the problem (20), then the minimizer M in equation (21) is the matrix C^TC where C is given by equation (16). Similarly, if $D = AB^T$, then $M = CC^T$.*

Proof. Let $D = B^TA$, and compare matrices M and C^TC. Recall that $C^TC = Y\Gamma^2Y^T$, where Γ^2 is an $r \times r$ diagonal matrix whose diagonal elements are the strictly positive eigenvalues of B^TA, and Y is an $n \times r$ matrix whose columns are the corresponding orthogonal left eigenvectors of B^TA. Then Γ^2 is formed from Λ_+ by deleting the $n-r$ rows and columns consisting only zeros. Since B^TA is symmetric, its right eigenvectors are the same as the left eigenvectors. The r columns of Y and the firs t r columns of Z corresponding to strictly positive eigenvalues are the same. The remaining $n - r$ columns of Z vanish in the multiplication since the corresponding $n - r$ rows and columns of Λ_+ are zero. Hence $M = C^TC$.

In the same way we get that $M = CC^T$ if $D = AB^T$. □

4. Examples

In this section we compute in simple examples the Bonacich measures and matrices C and M of equations (16) and (21), respectively.

Example 1. Let A and B be given by

$$A = \begin{bmatrix} 1/4 & 3/4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{22}$$

Then B^TA and AB^T are the following (row) stochastic matrices

$$B^TA = \begin{bmatrix} 0 & 1 \\ 1/4 & 3/4 \end{bmatrix}, \quad AB^T = \begin{bmatrix} 3/4 & 1/4 \\ 1 & 0 \end{bmatrix} \tag{23}$$

The greatest eigenvalue λ_1 of these matrices is 1, and the corresponding left eigenvectors (with norm 1, and as row vectors) are $y^1 = (1/\sqrt{17}, 4/\sqrt{17})$ (matrix B^TA) and $x^1 = (4/\sqrt{17}, 1/\sqrt{17})$ (matrix AB^T). These are the (inflow) Bonacich measures of the directed bimodal network. The right eigenvectors corresponding to $\lambda_1 = 1$ are $(\sqrt{2}/2, \sqrt{2}/2)$ for both matrices, and these would be the outflow Bonacich measures.

The second eigenvalue of $B^T A$ and AB^T is $\lambda_2 = -1/4$. The left eigenvector of $B^T A$ corresponding to $\lambda_2 = -1/4$ is $y^2 = (-\sqrt{2}/2, \sqrt{2}/2)$, and the left eigenvector of AB^T is $x^2 = (\sqrt{2}/2, -\sqrt{2}/2)$. The right eigenvector of $B^T A$ corresponding to $\lambda_2 = -1/4$ is $(-4/\sqrt{17}, 1/\sqrt{17})$, and the right eigenvectors of AB^T is $(1/\sqrt{17}, -4/\sqrt{17})$.

Equation (16) becomes

$$C = \begin{pmatrix} 4/\sqrt{17} \\ 1/\sqrt{17} \end{pmatrix} \cdot 1 \cdot \left(1/\sqrt{17}, \quad 4/\sqrt{17}\right) = \frac{1}{17}\begin{bmatrix} 4 & 16 \\ 1 & 4 \end{bmatrix} \tag{24}$$

Hence we have

$$C^T C = \frac{1}{17}\begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}, \quad CC^T = \frac{1}{17}\begin{bmatrix} 16 & 4 \\ 4 & 1 \end{bmatrix} \tag{25}$$

The symmetric part of $B^T A$ is

$$\frac{1}{2}\left(B^T A + A^T B\right) = \frac{1}{8}\begin{bmatrix} 0 & 5 \\ 5 & 6 \end{bmatrix} \tag{26}$$

The greatest eigenvalue of this matrix is $\mu = (3 + 4\sqrt{2})/8$, and the corresponding eigenvector (with norm one, and as a row vector) is z^1. Denote the eigenvector orthogonal to z^1 by z^2. These vectors can be written as

$$z^1 = \frac{1}{\sqrt{6(11 + 4\sqrt{2})}}\left(5, 3 + 4\sqrt{2}\right), \quad z^2 = \frac{1}{\sqrt{6(11 + 4\sqrt{2})}}\left(3 + 4\sqrt{2}, -5\right).$$

When $D = B^T A$, we can express the matrix M in equation (21) by

$$M = \frac{1}{6(11 + 4\sqrt{2})}\begin{bmatrix} 5 & 3 + 4\sqrt{2} \\ 3 + 4\sqrt{2} & -5 \end{bmatrix}\begin{bmatrix} \frac{3+4\sqrt{2}}{8} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 3 + 4\sqrt{2} \\ 3 + 4\sqrt{2} & -5 \end{bmatrix} \tag{27}$$

Let $\tau = 8\mu = 3 + 4\sqrt{2}$, and equation (27) simplifies to

$$M = \frac{\tau}{48(8 + \tau)}\begin{bmatrix} 25 & 5\tau \\ 5\tau & \tau^2 \end{bmatrix} \tag{28}$$

Example 2. Let A and B be given by

$$A = \begin{bmatrix} 1/4 & 3/4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3/4 & 1/4 \\ 1 & 0 \end{bmatrix} \tag{29}$$

Then $B^T A$ and AB^T are the following matrices

$$B^T A = \frac{1}{16}\begin{bmatrix} 3 & 25 \\ 1 & 3 \end{bmatrix}, \quad AB^T = \frac{1}{16}\begin{bmatrix} 6 & 4 \\ 4 & 0 \end{bmatrix} \tag{30}$$

The greatest eigenvalue of both $B^T A$ and AB^T is $\lambda_1 = 1/2$, and the corresponding left eigenvectors (with norm 1, and as row vectors) are $y^1 = (1/\sqrt{82}, 9/\sqrt{82})$ (matrix $B^T A$) and

$x^1 = (2/\sqrt{5}, 1/\sqrt{5})$ (matrix AB^T). These are the (inflow) Bonacich measures of the directed bimodal network. The right eigenvectors corresponding to $\lambda_1 = 1$ are $(5/\sqrt{26}, 1/\sqrt{26})$ for $B^T A$, and the right eigenvector of AB^T is $x^1 = (2/\sqrt{5}, 1/\sqrt{5})$. These vectors are the outflow Bonacich measures.

The second eigenvalue of $B^T A$ and AB^T is $\lambda_2 = -1/8$. The left eigenvector of $B^T A$ corresponding to $\lambda_2 = -1/8$ is $y^2 = (-1/\sqrt{26}, 5/\sqrt{26})$, and the left eigenvector of AB^T is $x^2 = (-1, 0)$. The right eigenvector of $B^T A$ corresponding to $\lambda_2 = -1/8$ is $(5/\sqrt{26}, -11/\sqrt{26})$, and the right eigenvectors of AB^T is $x^2 = (-1, 0)$.

Equation (16) becomes

$$C = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \cdot \frac{\sqrt{2}}{2} \cdot \left(1/\sqrt{82},\ \ 9/\sqrt{82}\right) = \frac{\sqrt{820}}{820} \begin{bmatrix} 2 & 18 \\ 1 & 9 \end{bmatrix} \tag{31}$$

Hence we have

$$C^T C = \frac{1}{164} \begin{bmatrix} 1 & 9 \\ 9 & 81 \end{bmatrix}, \qquad CC^T = \frac{1}{67240} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \tag{32}$$

The symmetric part of $B^T A$ is

$$\frac{1}{2}\left(B^T A + A^T B\right) = \frac{1}{16} \begin{bmatrix} 3 & 13 \\ 13 & 3 \end{bmatrix} \tag{33}$$

The greatest eigenvalue of this doubly stochastic matrix is $\mu = 1$, and the corresponding eigenvector is $z^1 = (\sqrt{2}, \sqrt{2})/2$. The eigenvector orthogonal to z^1 is $z^2 = (\sqrt{2}, -\sqrt{2})/2$. When $D = B^T A$, we can express the matrix M in equation (21) by

$$M = \frac{1}{4} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{34}$$

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